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ACKNOWLEDGMENT

The editors of the Mathematics Magazine take this opportunity to express our appreciation to Dr. Glenn James, the retiring managing editor, for the service which he has rendered to the mathematical world in guiding the activities of the Mathematics Magazine for the past twelve years. As a result of his policies and leadership, the Mathematics Magazine has grown in scope and circulation. It has come to fill a definite need in the area of collegiate mathematics by providing carefully prepared expository articles of interest to both the college teacher of mathematics and the advanced mathematics student. Mathematical research has benefited from the publication of research papers and notes in the Mathematics Magazine. Aspects of mathematics teaching have been discussed in an effort to aid in improved educational programs. General interest in mathematics has been promoted through the discussion of current books and the presentation of mathematical problems. All of these activities have attained their present status in the Magazine during the tenure of Dr. James as managing editor.

The readers of the Mathematics Magazine are also deeply indebted to Mrs. Inez James for her service as circulation manager and correspondence secretary for the past twelve years. This fine service has always been given without cost to the Magazine. Mrs. James is also retiring from this service at this time.

Dr. James was born in Wabash, Indiana, and attended school in Jonesboro and Monticello, Indiana. He did his undergraduate work in mathematics at Vincennes and Indiana Universities. He did graduate study at Indiana and Chicago Universities and earned the Ph. D. degree in mathematics at Columbia University in the year 1917. Dr. James taught in the mathematics departments of Michigan State University, Purdue University, Carnegie Institute of Technology, and for many years until his retirement from teaching was a member of the mathematics faculty at the University of California at Los Angeles. His major mathematical interests lie in the studies of infinite series and number theory, especially the famous last theorem of Fermat. He published numerous research papers, with his son R. C. James, edited the Mathematics Dictionary, and edited The Tree of Mathematics in addition to his work on the Mathematics Magazine.

We are sure the many friends and colleagues of Dr. James and Mrs. James as well as the readers of the Mathematics Magazine join us in extending our thanks for many tasks well done and our best wishes for the future at this time of retirement.

Finally, we affirm our intention to continue the development of the Mathematics Magazine as a contributor to progress in the mathematical world.

The Editors

A MATRIC GENERAL SOLUTION OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

Ralph W. Pfouts and C. E. Ferguson

Linear difference equations with constant coefficients have been used for many years in various disciplines, and a general method of solution has become well known. In this paper an alternative general solution is presented; a statement of the linear difference equations system in terms of linear vector spaces then follows naturally from the alternative solution. The special case of the second-order linear difference equation is considered first, not merely because it is a simple case but also because it is a form which has many applications in the social sciences, especially in economics and psychology.

1. *The Second-Order Linear Difference Equation.*—The homogeneous portion of the second-order linear difference equation

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + F(t) \tag{1}$$

may be shown as

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \end{bmatrix}. \tag{2}$$

Equation (2) clearly shows the recursive nature of a difference equation system. However, all of the essential features of the system are retained if (2) is abbreviated to read

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix}. \tag{3}$$

Using R to designate the square matrix in (3), we note that the characteristic equation of R is

$$|R-\lambda I| = \lambda^2 - a_1\lambda - a_2 = 0 \ , \tag{4}$$

which is also the reduced characteristic equation of the square matrix in (2). It is obvious that the right-hand member of (4) is also the auxiliary equation of

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} \ . \tag{5}$$

Since the trace of a matrix equals the sum of the characteristic roots, and since the determinant of a matrix is the product of the characteristic

roots, we may rewrite (3) as

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_1\lambda_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix}. \tag{6}$$

Taking advantage of the recursive nature of (3), we write the general homogeneous solution as

$$\begin{bmatrix} y_t \\ y_{y-1} \end{bmatrix} = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_1\lambda_2 \\ 1 & 0 \end{bmatrix}^{t-1} \begin{bmatrix} y_1 \\ y_1 \end{bmatrix}, \tag{7}$$

where y_1 and y_0 are the initial values of the dependent variable expressed as deviations from the value of the particular integral. Then if $\overline{Y}_t$ is a particular integral of (1), a general solution of the equation is

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} \lambda_1+\lambda_2 & -\lambda_1\lambda_2 \\ 1 & 0 \end{bmatrix}^{t-1} \begin{bmatrix} Y_1 \\ Y_0 \end{bmatrix} + \begin{bmatrix} \overline{Y}_t \\ \overline{Y}_{t-1} \end{bmatrix}, \tag{8}$$

where Y_1 and Y_0 are the initial conditions. Y_t is shown as a function of t in (8).

A question of importance in applications is whether (1) is stable, in the sense of economics, or damped, in the sense of physics. That is, we wish to ask whether Y_t converges to $\overline{Y}_t$. It is clear from (8) that the solution is stable if, and only if, R converges to the null matrix. This, in turn, can occur if, and only if, the λ's are less than unity in absolute value, a point which is proved in the next section.

Frequently it is desirable to determine stability from the coefficients of (1). In the second-order case, the necessary and sufficient conditions are

$$\begin{aligned} &1-|a_2| > 0\,, \\ &1-a_1-a_2 > 0\,, \\ &1+a_1-a_2 > 0\,. \end{aligned} \tag{9}$$

These conditions are well known, cf. for example [6].

2. *The General Linear Difference Equation.* – The matric general solution of the general linear difference equation

$$Y_t = \sum_{i=1}^{n} a_i Y_{t-i} + F(t) \tag{10}$$

may be found in a manner analogous to that used for the second-order equation. Thus the homogeneous portion of (10) may be exhibited as

$$(11)\qquad \begin{bmatrix} y_t \\ y_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{t-n+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_{n-t} \end{bmatrix}.$$

Using R to designate the square matrix in (11), it follows immediately that

$$(12)\qquad |R - \lambda I| = (-\lambda)^n + a_1(-\lambda)^{n-1} + \cdots + a_{n-1}(-\lambda) + a_n .$$

If one sets the expression of the right in (12) equal to zero, it is clearly the auxiliary equation of (10). Let us call this expression equation (12a). From the elementary theory of equations, we recall that the coefficient a_i is equal to the sum of the combinations of the n roots of (12a) taken i at a time. Thus we write

$$(13)\qquad S_i = \sum_{C_i^n} \lambda_j \lambda_k \cdots \lambda_r$$

to indicate that i roots are included in each product and that there are $\binom{n}{i}$ terms in the summation.

Then the solution of the homogeneous system (11) is

$$(14)\qquad \begin{bmatrix} y_t \\ y_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ y_{t-n+1} \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & \cdots & S_n \\ & & & 0 \\ & I_{n-1} & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & 0 \end{bmatrix}^{t-n+1} \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ y_0 \end{bmatrix},$$

where I is the unit matrix and $y_0, \cdots, y_{n-1}$ are the initial values of the homogeneous variable. Similarly, if $\overline{Y}_t$ is a particular integral of (10), the general solution of (10) is

$$(15)\qquad \begin{bmatrix} Y_t \\ Y_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ Y_{t-n+1} \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & \cdots & S_n \\ & & & 0 \\ & I_{n-1} & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & 0 \end{bmatrix}^{t-n+1} \begin{bmatrix} Y_{n-1} \\ Y_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ Y_0 \end{bmatrix} + \begin{bmatrix} \overline{Y}_t \\ \overline{Y}_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ \overline{Y}_{t-n+1} \end{bmatrix}.$$

Let us turn again to the question of stability. We first prove a necessary

and sufficient condition which depends upon the characteristic roots of R. Next we present one necessary and three sufficient conditions; these conditions are developed directly from the elements of R. Finally we state the so-called Schur-Cohn conditions which are both necessary and sufficient for stability. These conditions also involve only the elements of R. However numerical values of the elements must be known if the Schur-Cohn conditions are to be useful.

Theorem: A necessary and sufficient condition for stability of the difference system (14) is that the characteristic roots of R be less than unity in absolute value.

The proof of this theorem is well known cf., for example Mirsky, [4, pp. 328-29]. But we can sketch a simpler proof using the information at hand.[1] In equations (3) and (14) the full difference systems are abbreviated so that the minimum and characteristic polynomials of R are equal. Hence R is nonderogatory and, for this reason, similar to the Jordan canonical form

$$J = \Lambda + C \,, \tag{16}$$

where Λ is an $(n \times n)$ diagonal matrix whose elements are the characteristic roots of R and C is an $(n \times n)$ matrix with ones on the first superdiagonal and zeros elsewhere. Consequently R^s converges to the null matrix if J^s does so. Finally, if R^s converges to the null matrix as s increases without bound, the solution of the difference system converges to the particular integral.

Consider the binomial expansion of J^s for $s > n$:

$$J^s = \Lambda^s + s\Lambda^{s-1}C + \binom{s}{2}\Lambda^{s-2}C^2 + \cdots + \binom{s}{n-1}\Lambda^{s-n+1}C^{n-1} \,. \tag{17}$$

For any integer which equals or exceeds n, the power series expansion contains exactly n terms since C^n is the null matrix. Without loss of generality we may assume that $\Lambda = \lambda I$. Substituting this expression in (17) yields

$$J^s = \lambda^s I + s\lambda^{s-1}C + \binom{s}{2}\lambda^{s-2}C^2 + \cdots + \binom{s}{n-1}\lambda^{s-n+1}C^{n-1} \,. \tag{18}$$

To prove sufficiency, we assume $|\lambda| < 1$. As s increases without bound, $\lambda^s \to 0$; thus the main diagonal of J^s contains only zeros. The remaining terms in (18) give the entries on the successive superdiagonals of J^s, namely $\binom{s}{k}\lambda^{s-k}$ for the k-th superdiagonal. Each of these terms goes to zero as s increases without bound. This is seen by considering the general term $\binom{s}{k}\lambda^{s-k}$ as a sequence of numbers generated by increasing the value of s. It is well understood that this sequence converges to zero as s increases without bound. Hence all other elements of J^s approach zero, and J^s converges to the null matrix.

Necessity is immediately proved by assuming $|\lambda| \geqq 1$. Then $|\lambda|^s \geqq 1$ and the first term in (18) does not vanish. Accordingly J^s does not converge to the null matrix. This completes the proof for the Jordan canonical form.

Since R is similar to J, it also proves the same property for R^s.

Necessary conditions for stability involving the coefficients of (12) may be obtained from (13) by noting that if the λ's are of modulus less than unity, each term in the summation in (13) will be less than unity, and consequently

$$|S_i| < \binom{n}{i} . \quad (i = 1, 2, \cdots, n) \tag{19}$$

The inequalities of (19) are necessary conditions for stability.

Sufficient conditions for stability have been developed by several different writers. Some of these conditions may be easily shown. The left-hand side of the determinental equation in (12a) may be written as a function of λ :

$$f(\lambda) = \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0 . \tag{20}$$

The zeros, or roots, of this polynomial occur, of course, at the points at which the polynomial is itself equal to zero.

As early as 1799 Gauss [2] showed that the polynomial in (20) has no zeros outside of the circle $|\lambda| = r$, where

$$r = \max\,(1, \sqrt{2}\, S),$$

and S is the sum of the positive C_j coefficients. For stability the roots must lie within the unit circle. Consequently, the Gauss test is not directly relevant since the minimum possible value of r is unity. That is, one or more roots could lie on the unit circle, giving rise neither to explosive movements nor damping. However, the Gauss test does show that the sum of the positive coefficients in the characteristic equation must be less than $2^{-1/2}$ for stability.

A somewhat similar, but more stringent, test was constructed by Walsh [8]. He showed that the roots of (20) lie within the circle given by

$$|\lambda| = \sum_{j=1}^{n} |C_{n-j}|^{1/j} .$$

In particular, stability is assured if

$$\sum_{j=1}^{n} |C_{n-j}|^{1/j} \leqq 1 .$$

This provides a rapid test based upon all of the coefficients of (20).

A more easily applied test was developed by Kojima [3]. He proved that all of the roots of (20) lie within the circle $|\lambda| \leqq r$, where

$$r = \max \left[\frac{|a_0|}{|a_1|},\ 2\left|\frac{a_j}{a_{j+1}}\right| \right]$$

for $j = 1, 2, \cdots, n-1$. Using this test, the sufficient conditions for stability

may be quickly determined by scanning the coefficients to see if either of the coefficient ratios exceeds unity. A similar test was suggested by Montel [5], but it does not discriminate as finely.

We have now shown some necessary and some sufficient conditions. There exists one set of necessary and sufficient conditions which rely only upon the values of the elements of R. These conditions were originally developed by Schur [7]; they were later refined by Cohn and others, and since that time, the conditions have usually been called the Schur-Cohn conditions. Specifically, both writers proved that it is necessary and sufficient for stability that the matrix

$$(21) \qquad D_i = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_{n-i+1} \\ a_1 & 1 & \cdots & 0 & 0 & a_n & \cdots & a_{n-i+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1} & a_{i-2} & \cdots & 1 & 0 & 0 & \cdots & a_n \\ \hline a_n & 0 & \cdots & 0 & 1 & a_1 & \cdots & a_{i-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & 1 & \cdots & a_{i-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-i+1} & a_{n-i+2} & \cdots & a_n & 0 & 0 & \cdots & 1 \end{array}\right]$$

$(i = 1, 2, \cdots, n)$

be associated with a constrained positive definite quadratic form.

3. *The Difference Equation System in Linear Vector Space.*[2] The matric general solution of the linear difference equation system suggests, in a natural way, a statement of the linear difference equation in terms of linear vector spaces. To approach this subject we define

$$(22) \qquad V = \begin{bmatrix} y_i \\ y_{i-1} \\ \vdots \\ y_{i-n+1} \end{bmatrix}, \quad (i = n-1, \cdots, t)$$

where the V's are column vectors belonging to the polynomial in (12). Thus each V_i is an n-dimensional vector in the homogeneous variable of (10). Making use of the matrix R, we may show a sequence of these vectors as

$$(23) \qquad V_i,\ RV_i,\ R^2V_i,\ \cdots,\ R^{n-1}V_i.$$

The vectors of (23) are merely a consecutive sequence of n of the vectors defined in (22).

Any consecutive sequence of n vectors of (22) or (23) are linearly independent. This may be demonstrated by noting that the elements of any

vector in the sequence are related by the homogeneous difference equation

$$y_i = a_1 y_{i-1} + a_2 y_{i-2} + \cdots + a_n y_{i-n} ,$$

and the elements of any consecutive set of n vectors are similarly related. Consequently scalar multiplication of any one vector in a consecutive sequence means that all other vectors in the sequence are multiplied by the same scalar. Thus it is impossible to induce linear dependence in (23). From the definition of a homogeneous difference system, we observe that

$$R^n V_i = a_1 R^{n-1} V_i + a_2 R^{n-2} V_i + \cdots + a_n V_i . \tag{24}$$

Thus the $(n+1)$-st vector of a consecutive sequence is a linear combination of the preceding n vectors. Consequently a set of $(n+1)$ consecutive vectors is linearly dependent, but any set of n consecutive vectors is linearly independent.

The arguments just presented demonstrate that n consecutive vectors of (22) or (23) may be considered as a basis of a linear vector space Γ of dimension n. This space we call the solution space.

Now apply the theorem which states that any vector in the set spanning Γ may be expressed uniquely as a linear combination of vectors constituting a basis of Γ. Hence any solution vector V_r in Γ is expressible uniquely as a linear combination of consecutive vectors.

We now digress to prove the following

Lemma: If $W_1, W_2, \cdots, W_n$ denote the n linearly independent column vectors of a non-singular n-square matrix P, if A is an n-square matrix, and if $AW_j = b_{1j} W_{t-1} + b_{2j} W_{t-2} + \cdots + b_{nj} W_{t-n}$, then the column vector $[b_{1j}, b_{2j}, \cdots, b_{nj}]$ is the j-th column vector of $P^{-1}AP$.

By hypothesis the n vectors of P are linearly independent. Thus the vector AW_j, which is the j-th column vector of the matrix AP, may be expressed as stated in the lemma. If we denote the i-th row of P^{-1} by W^i, we have

$$W^i W_j = \delta_{t-i,j} ,$$

where $\delta_{t-i,j}$ is the Kronecker delta with respect to i and j. Then it follows immediately that

$$W^i A W_j = b_{t-i,j} .$$

This proves the lemma [1].

We return now to the chief argument. By definition of a homogeneous linear difference system

$$V_k = RV_{k-1} . \quad (k = n, \cdots, t) \tag{25}$$

From (24) and (25) we obtain

$$RV_t = a_1 V_{t-1} + a_2 V_{t-2} + \cdots + a_n V_{t-n} . \tag{26}$$

Now by virtue of the lemma proved above and (25) and (26), we see that a

vector V_k may be taken as a column of an n-square matrix P such that

$$(27) \qquad P^{-1}RP = R' = \begin{bmatrix} a_1 & & & \\ a_2 & & & \\ \cdot & & & \\ \vdots & & I_{n-1} & \\ a_n & 0 & \cdots & 0 \end{bmatrix},$$

where R' is the transpose of R.

It is apparent that

$$(28) \qquad P = \begin{bmatrix} y_t & y_{t-1} & \cdots & y_{t-n+1} \\ y_{t-1} & y_{t-2} & \cdots & y_{t-n} \\ \cdot & \cdot & & \cdot \\ \vdots & \vdots & & \vdots \\ y_{t-n+1} & y_{t-n} & \cdots & y_{t-2n+2} \end{bmatrix}.$$

Thus a set of n consecutive vectors in Γ may be used in forming a contragredient transform of R, and this transformation yields the transpose of R.

R and R' are rational canonical matrices. We now introduce the theorem which states that if A is an n-square matrix and $|A-\lambda I|$ has only one invariant factor different from unity, the matrix is similar to a rational canonical form. And under these conditions the invariant factor in question is the characteristic function of the rational canonical matrix [1, pp. 170-1].

As a consequence of the theorem stated in the preceding paragraph, the right member of (12a) is the only invariant factor of $|R-\lambda I|$ which is different from unity. Since (12a) is the auxiliary equation of the linear difference system, we may also say that the auxiliary equation is the only nontrivial invariant factor of the system. The auxiliary equation of a linear difference system is usually treated as if it were merely a convenient trick for obtaining the general solution. The arguments of this paper show that it is much more than this. Indeed it is fundamental to the difference equation system, for it is the only invariant factor different from unity and it also contains the constants associated with a basis of the linear vector solution space.

FOOTNOTES

1. The authors are indebted to Professor Leonard Carlitz of Duke University for suggesting this method of proof.

2. The argument of this section relies on that of [1] pp. 207-11.

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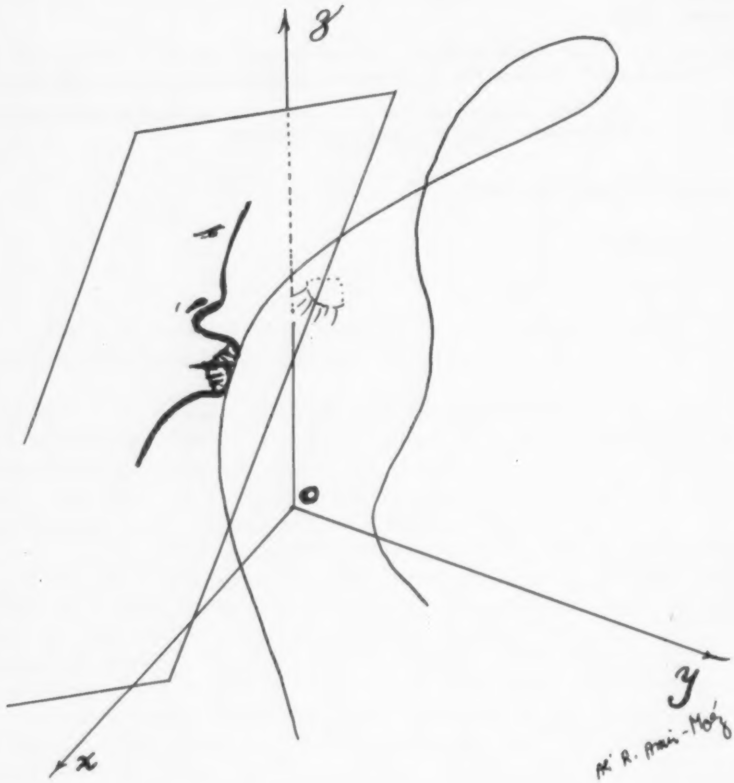

Osculating plane

ON UNEQUAL PARTITIONS OF INTEGERS

Iwao Sugai

SECTION I. INTRODUCTION

The ancient Greeks attached considerable religious and philosophical significance to numbers. Among all, Pythagoras highly valued square numbers such as 4, 9, 16 ⋯, he expressed the square of any positive integer, say i by the sum of i gnomons which increase their magnitudes in the order of appearance.[1] Thus, he partitioned i^2 into i distinct positive integers. This note is an extension of such an idea, i.e. for given positive integers say i and n, how many ways are there to express i^n by i distinct number elements, possibly including non-gnomons. Thus the method taken in this note does not resort to the well-developed theorems on prime numbers and congruences, but it uses extensively arithmetical progressions and finite differences. The scope of the general parition of integers under such methods is so vast that two conditions are imposed to facilitate study. They are discussed in the following sections.

NOTATION.

For positive integers i and n; $(i)^n$ is expressed as the sum of i distinct positive integers, such as

$$(i)^n = \sum_{m=1}^{m=i} A_{n,i,m} \quad \text{where } i = 1, 2, 3, \cdots, n, \cdots.$$

From these number elements an infinite "right-angled-number-triangle" is formed. If the diagonal elements and horizontal elements, or if the diagonal elements and vertical elements have zero n-th finite difference and $n \geqq 3$, the total possible number of such "number triangles" is at least $3 \cdot 2^{n-3}$. Thus $(i)^n$ has at least $3 \cdot 2^{n-3}$ partition types under the aforementioned conditions.

SECTION II.

CONDITIONS AND TWO INTERESTING TYPES OF PARTITIONS

Condition 1

For positive integers i and n, $(i)^n$ is expressed by the sum of i distinct positive integers, i.e.

$$(1) \qquad (i)^n = \sum_{m=1}^{m=i} A_{n,i,m} \quad \text{where } i = 1, 2, 3, \cdots, n, \cdots.$$

Eq. (1) is arranged as follows,

$$(1)^n = A_{n,1,1}$$

$$(2)^n = A_{n,2,1} + A_{n,2,2}$$

$$(3)^n = A_{n,3,1} + A_{n,3,2} + A_{n,3,3}$$

$$\vdots$$

$$(i)^n = A_{n,i,1} + A_{n,i,2} + A_{n,i,3} + \cdots + A_{n,i,p} + \cdots + A_{n,i,i}$$

$$(j)^n = A_{n,j,1} + A_{n,j,2} + A_{n,j,3} + \cdots + A_{n,j,p} + \cdots + A_{n,j,i} + A_{n,j,j}$$

$$\vdots$$

where $j = i+1$

Condition 2

In the diagram above the number elements must satisfy one of the following conditions:

(a) Diagonal elements $A_{n,i,i}$ and horizontal elements $A_{n,i,m}$ ($m = 1, 2, \cdots, i$) for all i's must have zero n-th finite differences.

or

(b) Diagonal elements $A_{n,i,i}$ and vertical elements $A_{n,i,p}$ ($p = 1, 2, \cdots, i$) for all i's must have zero n-th finite differences.

In order to handle many varieties of the infinite "right-angled-number-triangle" (as it appears above), three basic classifications are defined at this point.

TYPE (1)

(I) It is generated by the expression

$$(2) \qquad (i)^n = \sum_{m=1}^{m=i} [m^n - (m-1)^n] .$$

(II) Vertical elements $A_{n,i,p}$ are the same for all i's if p is held constant.

(III) Both diagonal and horizontal series are identical and they have $n!$ for the $(n-1)$th finite differences.

TYPE (2)

(I) It is generated by the expression

$$(3) \qquad (i)^n = \sum_{m=1}^{m=i} i^{(n-3)}[i^2 - i + 1 + 2(m-1)] .$$

(II) Number elements are in ever-increasing order, i.e. $A_{n,j,m} > A_{n,i,m}$ for any m, where $j = i+1$.

(III) Both diagonal and vertical elements have $(n-1)!$ for the $(n-1)$th finite differences.

TYPE (3)

Any other varieties of partition types which do not belong to either Type (1) or Type (2).

It is interesting to observe that the case of Pythagoras ($n = 2$) belongs to Type (1) and there are no other types. Thus this note deals with cases of $n \geqq 3$ to study many types of partitions of i^n.

SECTION III. GNOMONS FOR CUBE NUMBERS

The literature in the history of theory of numbers mentions often the gnomons for the square numbers, but there seems to be little mention of gnomons for cube numbers. Now gnomons for the cube numbers are illustrated by the following diagrams.

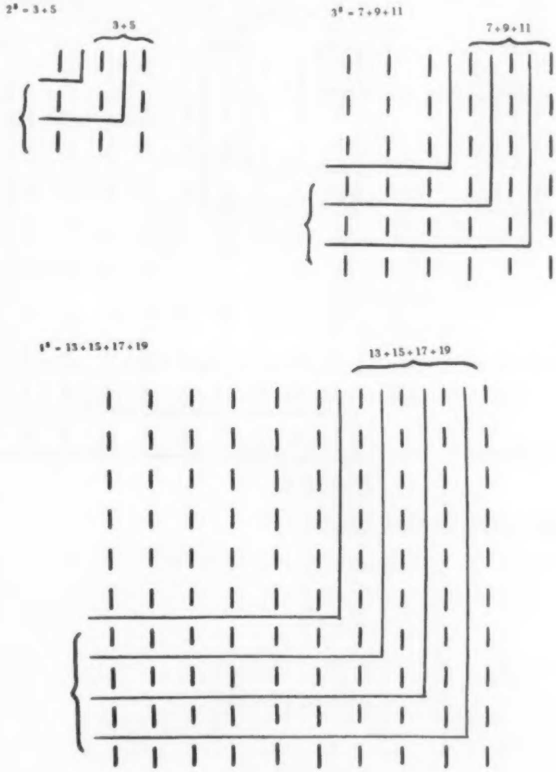

Figure 1

The number triangle consisting of these "successive" gnomons is represented below.

$$2^3 = 3 + 5$$

$$3^3 = 7+9+11$$

$$4^3 = 13+15+17+19$$

$$5^3 = 21+23+25+27+29$$

$$\vdots$$

$$i^3 = \sum_{m=s}^{m=s+i-1} (2m-1) \quad \text{where } s = 1 + \frac{i(i-1)}{2}$$

This was first reported in the literature by G. Polya.[2] Another type of representation by gnomons for the cube numbers is also shown here.

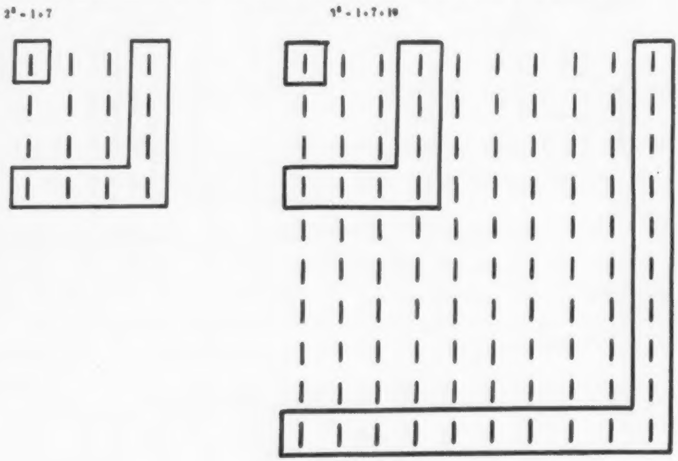

Figure 2

From these gnomons the number triangle

$$2^3 = 1+7$$

$$3^3 = 1+7+19$$

$$4^3 = 1+7+19+37$$

$$5^3 = 1+7+19+37+61$$

$$\vdots$$

is formed.

These represent two methods for dividing cube numbers into gnomons. As shown later, the "successive" gnomons are very useful in describing the Type (2) partition of this note. This idea of gnomons extends to any

higher powered numbers, so that i^n is expressed by the sum of i gnomons.

SECTION IV.

CONSTRUCTION OF VARIOUS TYPES OF PARTITION

(a) $n = 3$ *case*

The construction of three types of partitions is illustrated. Take $i = 2$ first. 2^3 must be expressed by the sum of two unequal positive integers. There are three ways to meet this requirement, namely $1+7$, $2+6$, and $3+5$. These three ways come from

$$2^3 = 2^2 \cdot 2 = (1+3)\cdot(1+1) .$$

by the "successive" gnomons
$3+5$

$$\begin{array}{ccc} 1 & + & 3 \\ 1 & + & 3 \\ \hline 1 & + & 7 \end{array} \qquad \begin{array}{c|cc} 1 & + & 3 \\ 1 & + & 3 \\ \hline 2 & + & 6 \end{array}$$

Next take $i = 3$, and notice that the ways the partitioning lines are drawn must be followed exactly.

$$3^3 = 3^2 \cdot 3 = (1+3+5)\cdot(1+1+1)$$

by the "successive" gnomons
$7+9+11$

$$\begin{array}{ccccc} 1 & + & 3 & + & 5 \\ 1 & + & 3 & + & 5 \\ 1 & + & 3 & + & 5 \\ \hline 1 & + & 7 & + & 19 \end{array} \qquad \begin{array}{c|cc|cc} 1 & + & 3 & + & 5 \\ 1 & + & 3 & + & 5 \\ 1 & + & 3 & + & 5 \\ \hline 3 & + & 9 & + & 15 \end{array}$$

A pattern similar to this sort of partition is shown in the book by Uspensky and Heaslet.[3] For all other i's the patterns are identical.

(b) $n = 4$ *case*

The construction of six types of partitions are explained in like manner.

$$2^4 = 2^3 \cdot 2 = \begin{Bmatrix} 1+7 \\ \text{or} \\ 2+6 \\ \text{or} \\ 3+5 \end{Bmatrix} \cdot (1+1)$$

$$\begin{array}{ccc} 1 & + & 7 \\ 1 & + & 7 \\ \hline 1 & + & 15 \end{array} \qquad \begin{array}{c|cc} 1 & + & 7 \\ 1 & + & 7 \\ \hline 2 & + & 14 \end{array} \qquad \begin{array}{c|cc} 2 & + & 6 \\ 2 & + & 6 \\ \hline 4 & + & 12 \end{array} \qquad \begin{array}{ccc} 2 & + & 6 \\ 2 & + & 6 \\ \hline 6 & + & 10 \end{array}$$

$$\begin{array}{ccc} 3 & + & 5 \\ 3 & + & 5 \\ \hline 3 & + & 13 \end{array} \qquad \begin{array}{ccc} 3 & + & 5 \\ 3 & + & 5 \\ \hline 5 & + & 11 \end{array}$$

At this stage it is useful to re-examine the generating function for

Type (2). It has the lowest power index with respect to the running index m, and Type (2) gives $2^4 = 6+10$. Therefore, any expression which gives $2^4 = 7+9$ does not involve m, i.e. by this generating function $(i)^n$ is not expressed by i distinct positive integers in general even though 2^4 is expressed by two positive integers. From the view point of gnomons for the fourth powered numbers, the partition type of $2^4 = 7+9$ is also rejected. Gnomons for the 2^4 are shown below:

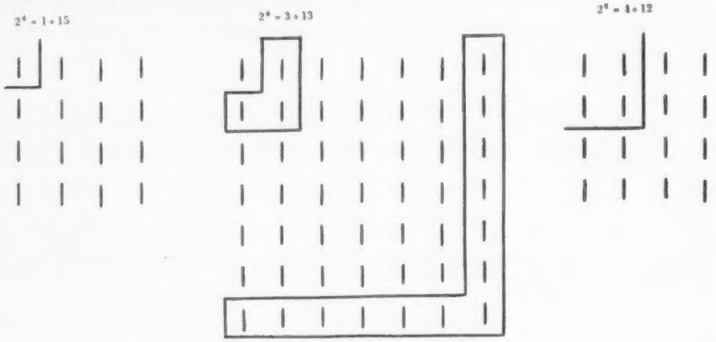

Figure 3

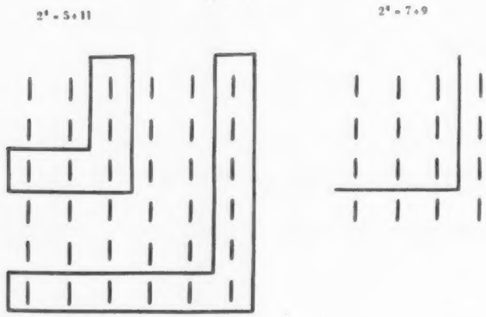

Figure 4

But the last case violates the nature of gnomons. Gnomons increase their magnitudes in the order of appearance.

Pythagoras' original gnomons are arrays of ones with width of a unit and his gnomons do not skip any array of ones, but some of our gnomons shown for $n = 3$ and $n = 4$ cases have width larger than a unit and our gnomons skip many arrays of ones. These may be called "generalized" gnomons.

SECTION V. THE FINAL RESULT

The result of this study is to show that the total number of unequal partitions of i^n, obeying Conditions (1) and (2) is at least $3 \cdot 2^{n-3}$.

For the case of $n = k$, k: any positive integer, $(i)^k$ is partitioned into

i distinct positive integers. The possible number for various partitions for i^k $(i \geqq 3)$ is equal to the possible number for various types of partitions of 2^k into two unequal positive integers, since the patterns for i^k $(i \geqq 3)$ are identical for the patterns for 2^k.

Now 2^k is partitioned into two positive integers from the highest power index of m, running index (Type 1) to the lowest power index of m, running index (Type 2).

$$\begin{array}{lll}
2^k = (1) + (2^k-1) & - & \text{Type 1} \\
\left.\begin{array}{l} 2^k = (2) + (2^k-2) \\ 2^k = (3) + (2^k-3) \end{array}\right\} & - & \text{Collection of Type 3's} \\
\vdots \quad \vdots \quad \vdots & & \\
2^k = (S) + (S+2\cdot 2^{k-3}) & - & \text{Type 2}
\end{array}$$

From the last line, S is solved to be $3\cdot 2^{k-3}$. Thus, there are at least $3\cdot 2^{k-3}$ types of partitions.

The auxiliary conditions for the n-th finite differences for diagonal plus horizontal or diagonal plus vertical number elements are obvious by proper application of calculus of finite differences [4] to various generating functions.

Of all these $3\cdot 2^{n-3}$ types of partition, the Type (1) has been applied for a practical purpose in extracting the n-th root of a positive integer on digital computers. [5]

SECTION VI. NUMERICAL EXAMPLES

Due to limited space, only the numerical examples of cases $n = 3$ and $n = 4$ are shown.

$n = 3$ *case*

$$\text{Type (1)} \qquad i^3 = \sum_{m=1}^{m=i} [m^3 - (m-1)^3]$$

$$\begin{array}{l}
1^3 = 1 = 1 \\
2^3 = 8 = 1 + 7 \\
3^3 = 27 = 1 + 7 + 19 \\
4^3 = 64 = 1 + 7 + 19 + 37 \\
6^3 = 216 = 1 + 7 + 19 + 37 + 61 + 91 \\
7^3 = 343 = 1 + 7 + 19 + 37 + 61 + 91 + 127 \\
\vdots \qquad\qquad\qquad\qquad \vdots
\end{array}$$

$$\text{Type (3)} \qquad i^3 = \sum_{m=1}^{m=i} i[m^2-(m-1)^2]$$

$$\begin{aligned}
1^3 &= 1\\
2^3 &= 2+6\\
3^3 &= 3+9+15\\
4^3 &= 4+12+20+28\\
5^3 &= 5+15+25+35+45\\
6^3 &= 6+18+30+42+54+66\\
7^3 &= 7+21+35+49+63+77+91\\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

$$\text{Type (2)} \qquad i^3 = \sum_{m=1}^{m=i} [i^2-i+1+2(m-1)]$$

$$\begin{aligned}
1^3 &= 1\\
2^3 &= 3+5\\
3^3 &= 7+9+11\\
4^3 &= 13+15+17+19\\
5^3 &= 21+23+25+27+29\\
6^3 &= 31+33+35+37+39+41\\
7^3 &= 43+45+47+49+51+53+55\\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

$n = 4$ *case*

$$\text{Type (1)} \qquad i^4 = \sum_{m=1}^{m=i} [m^4-(m-1)^n]$$

$$\begin{aligned}
1^4 &= 1 = 1\\
2^4 &= 16 = 1+15\\
3^4 &= 81 = 1+15+65\\
4^4 &= 256 = 1+15+65+175
\end{aligned}$$

$$5^4 = 625 = 1+15+65+175+369$$

$$\vdots \qquad\qquad \vdots$$

Type (3) – a $\quad i^4 = \sum_{m=1}^{m=i} i[m^3-(m-1)^3]$

$$1^4 = 1$$

$$2^4 = 2+14$$

$$3^4 = 3+21+57$$

$$4^4 = 4+28+76+148$$

$$5^4 = 5+35+95+285+305$$

$$\vdots \qquad\qquad \vdots$$

Type (3) – b $\quad i^4 = \sum_{m=1}^{m=i} [i(i-1)(2m-1)+(3m^2-3m+1)]$

$$1^4 = 1$$

$$2^4 = 3+13$$

$$3^4 = 7+25+49$$

$$4^4 = 13+43+79+121$$

$$5^4 = 21+67+119+177+241$$

$$\vdots \qquad\qquad \vdots$$

Type (3) – c $\quad i^4 = \sum_{m=1}^{m=i} [(2m-1)i^2]$

$$1^4 = 1$$

$$2^4 = 4+12$$

$$3^4 = 9+27+45$$

$$4^4 = 16+48+80+112$$

$$5^4 = 25+75+125+175+225$$

$$\vdots \qquad\qquad \vdots$$

Type (3) – d $$i^4 = \sum_{m=1}^{m=i} [i(i+1)(2m-1) - (3m^2-3m+1)]$$

$$1^4 = 1$$

$$2^4 = 5 + 11$$

$$3^4 = 11 + 29 + 41$$

$$4^4 = 19 + 53 + 81 + 103$$

$$5^4 = 29 + 83 + 131 + 173 + 209$$

$$\vdots \qquad \qquad \vdots$$

Type (2) $$i^4 = \sum_{m=1}^{m=i} i[i^2 - i + 1 + 2(m-1)]$$

$$1^4 = 1$$

$$2^4 = 6 + 10$$

$$3^4 = 21 + 27 + 33$$

$$4^4 = 52 + 60 + 68 + 76$$

$$5^4 = 105 + 115 + 125 + 135 + 145$$

$$\vdots \qquad \qquad \vdots$$

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ROCKERS AND ROLLERS

Gerson B. Robison

Some years ago, while picking up my small son's toy blocks, I became intrigued with the possibility of finding a cylindrical surface upon which a plank would roll in neutral equilibrium. The results appeared as a problem in the American Mathematical Monthly. (v. 59, Oct. 1952, p. 551) The solution is repeated below.

For some reason, the normal mathematical impulse to generalize lay dormant until recently, when it was stimulated by a proposed science exhibit at New Paltz Teachers' College. There is a straightforward approach to the problem which yields several interesting and constructible models. In this approach, the upper, moving surface, (the roller) is defined in polar co-ordinates, with the center of gravity at the pole, while the fixed surface, (the rocker), is defined by rectangular co-ordinates with the y-axis positive downward.

The requirement of neutral equilibrium means, first, that the center of gravity of the roller must travel in a horizontal path and, second, it must remain directly above the point of contact of the two curves in all positions. In addition, the roller must actually roll into each position. These conditions suggest the following formulation:

In Fig. 1 the roller curve is given by $r = r(s)$, $\theta = \theta(s)$. The pole is at $0'$, and the parameter s is the length of arc $A'P$. The rocker has for its x-axis the line of travel of the pole of the roller. Its curve is given by $x = x(s)$, $y = y(s)$, s here being the corresponding arc length AP, equal to $A'P$. We assume that all four functions are differentiable. In the figure, the initial ray and the origin have been chosen so that $x(0) = \theta(0) = 0$, but this is actually a boundary condition which may be modified in some cases.

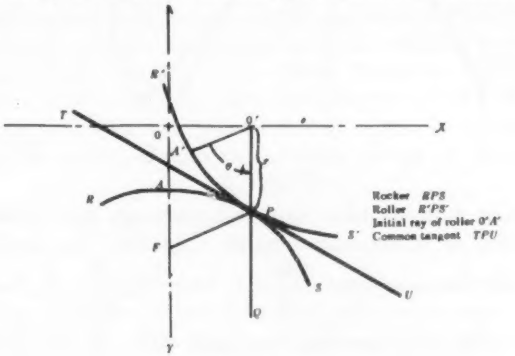


Figure 1

The condition of neutral equilibrium now gives us

(1) $$r(s) = y(s) \quad \text{for all values of } s$$

and the geometry yields

(2) $$\frac{dx}{dy} = \tan QPU = \frac{r\,d\theta}{dr}.$$

The special points where $\frac{dr}{ds} = \frac{dy}{ds} = 0$ do not give trouble in general.

To illustrate the method, let us start with the roller whose surface cross section is the straight line $r = \sec\theta$. By differentiation we get $\frac{d\theta}{dr} = 1/r\sqrt{r^2-1}$. Using (1) and (2) above we obtain $\frac{dx}{dy} = 1/\sqrt{y^2-1}$. So $x = \cosh^{-1}y + C$. We now take $\theta(0) = x(0) = 0$, which amounts to a correlating of the two coordinate systems. Then $r(0) = y(0) = 1$, and hence $C = 0$. We thus arrive at the equation $y = \cosh x$ for the rocker. (Fig. 2)

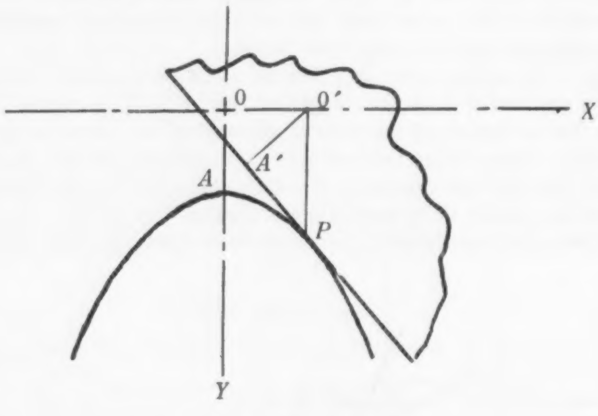


Figure 2

One can start with a rocker equation, although this seems to be less frequently fruitful of interesting results. Consider the family of oblique lines: $y = kx$. For these rockers $k\frac{dx}{dy} = 1$, whence $k\frac{d\theta}{dr} = \frac{1}{r}$, leading to $k\theta = \ln r + C$. In this case it is convenient to take $\theta(0) = 0$, but $\frac{1}{k}$ for $x(0)$. Then $y(0) = r(0) = 1$, again making $C = 0$. The roller family is then given by $r = e^{k\theta}$. (Fig 3) We have tabulated below some of the more interesting pairs. In each case the c.g. of the roller is at the pole of its equation. The last three columns correlate the two curves.

Number 5 has some special interest, for the two parabolas are actually

congruent. In fact, it can be shown by synthetic methods, using the focus-directrix definition of the parabola, that it is the only curve for which the rocker and roller can be congruent and in contact at corresponding points.

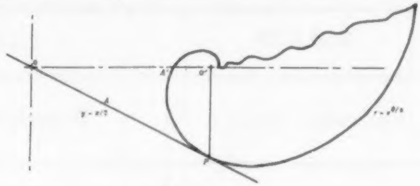


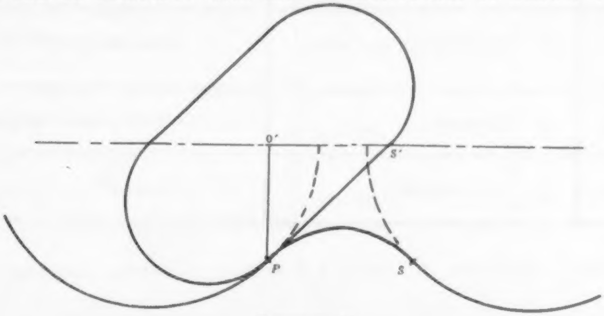
Figure 3

Referring to fig. 1, if F is the point of the rocker corresponding to $0'$ of the roller, then $FP = P0'$, and P is therefore on the parabola determined by $00'$ and F.

In actual construction of the models, some practical problems arise. Obviously, it is desirable to have surfaces with a high coefficient of friction, since the value of $\frac{dy}{dx}$ on the roller must not exceed this constant, in operation. A layer of fine sandpaper or of felt is effective, but allowance must be made for its thickness in the layout.

A thornier problem is that of getting the center of gravity where it belongs. The plank of number 1 is easy, and the parabola of number 5 will have its center of gravity properly located if it is a segment of altitude 1 2/3 . However, number 9 calls for a circular roller with the center of gravity on its circumference. One way of dealing with the problem is to double the roller. A system of two tangent circles has the centroid properlv placed.

In number 8 of the table, we find that a cardioid rolls on a cycloid. However, both curves have cusps, which means that there will be not only excessive slope, but also interference in the rolling action. By jumping from one lobe of the cardiod to the other along the common tangent, we obtain a roller which will roll on a combination of cycloid and catenary, the join taking place at the points of 60° slope. In a similar fashion, a catenary can be combined with the circular rocker of number 9 at the 45°



Catenary arc – PS
Figure 4

TABLE OF ROCKERS AND ROLLERS

	ROLLER	
Pair Number	Equation : $r =$	Description
1	$\sec\theta$	straight line
2	$e^{k\theta}$	equiangular spirals
3	$\dfrac{1+\epsilon}{1+\epsilon\cos\theta}$ $(\epsilon>1)$	hyperbolas
4	$\dfrac{1+\epsilon}{1+\epsilon\cos\theta}$ $(0<\epsilon<1)$	ellipses
5	$\dfrac{2}{1+\cos\theta}$	parabola
6	θ^k	spiral
7	θ^{-1}	hyperbolic spiral
8	$1-k\cos\theta$	limacon (cardioid if $k=1$)
9	$2\cos\theta$	circle, unit radius
10	$r=\cos n\theta$	"roses"

ROCKER				
Equation: $y =$	Description	$x(0)$	$\theta(0)$	$r(0)$
$\cosh x$	catenary	0	0	1
kx	oblique lines	$\frac{1}{k}$	0	1
$\frac{1}{\epsilon-1}(\epsilon\cosh\sqrt{\frac{\epsilon-1}{\epsilon+1}}\,x-1)$	catenary-like	0	0	1
$\frac{1}{1-\epsilon}(1-\epsilon\cos\sqrt{\frac{1-\epsilon}{1+\epsilon}}\,x)$	sinusoidal	0	0	1
$1+\frac{x^2}{4}$	parabola	0	0	1
$((1+k)x)^{k/1+k}$ $(x>0$ for $k>0)$ $(x>0$ for $-1<k<0)$ $(x<0$ for $k<-1)$	parabola-like hyperbola-like parabola-like*	$\frac{1}{1+k}$	1	1
e^{-x}	exponential	0	1	1
$x = t-k\sin t$ $y = 1-k\cos t$	trochoid (cycloid)	0	0	$1-k$
$x^2+y^2 = 4$	circle, radius 2*	0	0	2
$n^2x^2+y^2 = 1$	ellipses*	0	0	1

* Rocker concave upward in number 9, number 10, and third species of number 6.

points, for a roller consisting of two tangent circles and their common external tangents. (Fig. 4)

We conclude with a comment on parametric forms. If θ, x, and y ($= r$) are differentiable functions of any parameter t, then (1) can be reduced to $\frac{dx}{dt} = r\frac{d\theta}{dt}$. Hence, if the roller is given parametrically, the above equation may be used to obtain $x(t)$, $y(t)$ being given with $r(t)$. In particular, we may take θ as the parameter, whereupon the foregoing equation becomes simply $\frac{dx}{dt} = r$. Number 8 is an illustration of this. Conversely, given parametric equations for the rocker, to obtain parametric representation of the roller we may use $\frac{d\theta}{dt} = \frac{1}{y}\cdot\frac{dx}{dt}$.

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TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

USE OF MATRICES IN TEACHING CONIC SECTIONS

A. R. Amir-Moéz

Algebra as a tool has been used in problems of geometry. But we are always far behind in using a new tool when discovered. There is little hesitation in using determinants in elementary algebra. I don't see why we should hesitate to use matrices in freshmen analytic geometry. I believe it brings a new interest and also it eliminates some tedious arithmetic.

1. Definition: We are used to writing a point of the plane as $P = (x, y)$. Let us ignore " ," and write it as $(x\ y)$. We call $(x\ y)$ a row matrix corresponding to the point P. To the point P also corresponds $\binom{x}{y}$, called a column matrix. Now we generalize this idea as follows:

An array of numbers, for example

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -2 \end{pmatrix},$$

is called a matrix of two rows and three columns. In what follows we do not use matrices having more than three rows or columns.

It is custommary to write a matrix as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Two subscripts are used to indicate to which row and which column an element belongs. For example a_{23} is the element located in the second row and in the third column.

Two matrices are equal if and only if all corresponding elements are equal.

2. Addition of matrices: We define addition only for matrices which have the same number of rows and columns. The example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}$$

shows how addition is defined, i.e., corresponding elements have been added.

3. Multiplication of a number and a matrix: To multiply a number p by a matrix we multiply every element by p, as in the following example

$$p\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}p = \begin{pmatrix} pa_{11} & pa_{12} \\ pa_{21} & pa_{22} \end{pmatrix}.$$

4. Product of two matrices: Two matrices can be multiplied if the number of columns in the first one is the same as the number of rows in the second. We shall demonstrate the multiplication by the following example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ a_{31}b_{11}+a_{32}b_{21} & a_{31}b_{12}+a_{32}b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}.$$

We see that, for example,

$$c_{31} = a_{31}b_{11} + a_{32}b_{21}$$

is obtained by adding the product of the first element of the third row of the first matrix and the first element of the first column of the second matrix to the product of the second element of the third row of the first matrix and the second element of the first column of the second matrix.

It is important to observe that the operation of taking the product of two matrices is not necessarily commutative. For example

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -5 & -6 \\ -4 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ -5 & -3 \end{pmatrix}.$$

For convenience we shall denote a matrix by a capital letter such as $A = \begin{pmatrix} 2 & 1 \\ 7 & 0 \end{pmatrix}$.

For matrices A, B, and C the reader may verify that $A(B+C) = AB + AC$ if the multiplication and addition are possible.

5. Transpose of a matrix: If we interchange rows and columns of a matrix A we call the new matrix the transpose of A and we denote it by A'. For example if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ then } A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

It is easy to show that $(AB)' = B'A'$, (note the order). We leave it to the reader to verify this property. We also observe that $(A+B)' = A' + B'$.

6. Straight line: An equation of a straight line which passes through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = [(y_2 - y_1)/(x_2 - x_1)](x - x_1).$$

This equation can be written as

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = t,$$

where t is the common value of these fractions. Sometimes it is desirable to write these equations as

$$\begin{aligned} x - x_1 &= t(x_2 - x_1) = tl \\ y - y_1 &= t(y_2 - y_1) = tm \end{aligned} \tag{1}$$

which are called parametric equations of the line and l and m are called direction numbers of the line. Now we write the matrix equation of the line which is essentially the same as (1);

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} l \\ m \end{pmatrix}, \text{ or}$$

$$(x \quad y) = (x_1 \quad y_1) + t(l \quad m).$$

Now let $X = (x \quad y)$, $X_1 = (x_1 \quad y_1)$, and $D = (l \quad m)$. Then (1) can be written as

$$X = X_1 + tD, \text{ or } X' = X_1' + tD'.$$

7. Conic sections: An equation of the second degree in two variables is a conic section and can be written in the form

$$ax^2 + 2bxy + cy^2 + 2px + 2qy + r = 0.$$

This equation can be written using matrix multiplication in the form

$$(x \quad y \quad 1) \begin{pmatrix} a & b & p \\ b & c & q \\ p & q & r \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (0). \tag{1}$$

The reader may multiply and check the equation. Now let $X = (x \quad y \quad 1)$,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \text{ and } \quad Q = \begin{pmatrix} a & b & p \\ b & c & q \\ p & q & r \end{pmatrix}.$$

Then (1) can be written as

$$XQX' = 0 .$$

8. Intersection of a line and a conic: Let $XQX' = 0$ be a conic section and let $X = X_1 + tD$ be a straight line. Here $X = X_1 + tD$ is taken to be

$$(x \quad y \quad 1) = (x_1 \quad y_1 \quad 1) + t(l \quad m \quad 0) .$$

The points of intersection are the solutions of the system of equations

$$\begin{cases} XQX' = 0 \\ X = X_1 + tD . \end{cases}$$

Setting $X = X_1 + tD$ in the first equation we get

(1) $\quad (X_1 + tD)Q(X_1' + tD') = 0$, or $X_1QX_1' + tX_1QD' + tDQX_1' + t^2 DQD' = 0$.

(We have to keep in mind that multiplication is not commutative). We leave it to the reader to show that

$$X_1QD' = DQX_1' .$$

Thus (1) may be written as

(2) $$(DQD')t^2 + 2(X_1QD')t + X_1QX_1' = 0 .$$

We see that (DQD'), (X_1QD'), and X_1QX_1' are numbers (matrices of one element). We also note that to a point we can make correspond either $(x \quad y)$ or $(x \quad y \quad 1)$. We shall denote these row matrices by X and $D = (l \quad m)$ or $(l \quad m \quad 0)$. We shall use them where suitable.

Let us study the following:

$$DQD' = (l \quad m \quad 0)\begin{pmatrix} a & b & p \\ b & c & q \\ p & q & r \end{pmatrix}\begin{pmatrix} l \\ m \\ 0 \end{pmatrix} = (l \quad m)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} l \\ m \end{pmatrix} = DAD' .$$

Thus (2) can be written as

(3) $$(DAD')t^2 + 2(X_1QD')t + X_1QX_1' = 0 .$$

Now the values of t obtained from (3) can be substituted into $X = X_1 + tD$ and we get the points of intersection.

9. Discussion of the intersection:

I. If $DAD' \neq 0$, then the line intersects the conic in two points, real or complex conjugate, or coincident. This is determined through the discriminant of [8.(3)].

II. If $DAD' = 0$, $X_1QD' \neq 0$, the line and conic intersect in one point.

III. If $DAD' = 0$, $X_1QD' = 0$, but $X_1QX_1' \neq 0$, the line does not intersect

the conic.

IV. If [8.(3)] is an identity, then the line is actually a part of the conic.

10. Center of a conic : A point (x_0, y_0) is called a center of a conic if for any point M of the conic, there is another point N on the conic such that P is the midpoint of the segment MN.

Now to find the center let $P = (x_0 \;\; y_0 \;\; 1)$ and $M = (x \;\; y \;\; 1)$. The equation of the line through P and M is

(1) $$X = P + tD,$$

where $D = (x - x_0 \;\; y - y_0 \;\; 0)$ [see 6.]. As in [8.] the points of intersection of (1) and the conic $XQX' = 0$ are found by

(2) $$(DAD')t^2 + 2(PQD')t + PQP' = 0.$$

Let $DAD' \neq 0$; otherwise there will not be two points of intersection. If t_1 and t_2 are the two roots of (2), then the two points of intersection will be

$$M = (x, y) = (x_0 + t_1[x - x_0], y_0 + t_1[y - y_0])$$

and

$$N = (x_1, y_1) = (x_0 + t_2[x - x_0], y_0 + t_2[y - y_0]).$$

Thus

$$P = (x_0, y_0) = (x_0 + [x - x_0]\frac{t_1 + t_2}{2}, y_0 + [y - y_0]\frac{t_1 + t_2}{2}).$$

This implies that $\frac{t_1 + t_2}{2} = 0$. But $t_1 + t_2$ is the sum of the roots of (2) and we have

$$\frac{QPD'}{DAD'} = 0.$$

Since $DAD' \neq 0$ the equation $PQD' = 0$ gives the centers if there are any. That is

$$(x_0 \;\; y_0 \;\; 1)\begin{pmatrix} a & b & p \\ b & c & q \\ p & q & r \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \\ 0 \end{pmatrix} = 0,$$

or

$$(ax_0 + by_0 + p)(x - x_0) + (bx_0 + cy_0 + q)(y - y_0) = 0.$$

Now two cases may be considered :

I. If $(x, y) = (x_0, y_0)$, then every point of the conic is a center and the conic is a double line;

II. If $(x, y) \neq (x_0, y_0)$, then we must have

$$\begin{cases} ax_0 + by_0 + p = 0 \\ bx_0 + cy_0 + q = 0, \end{cases}$$

or

$$(x_0 \quad y_0 \quad 1)\begin{pmatrix} a & b \\ b & c \\ p & q \end{pmatrix} = 0.$$

This system of equations will give the centers, or we might write it as

$$(3) \qquad \begin{cases} ax + by + p = 0 \\ bx + cy + q = 0 . \end{cases}$$

11. Discussion of the center: The pair of equations [10.(3)] can be considered as a system giving the point of intersection of two lines.

I. If $b^2 - ac = 0$ and $aq - pb \neq 0$, then the two lines of [10.(3)] are parallel and they do not intersect. Thus the conic does not have a center (parabola has no center).

II. If $b^2 - ac = 0$ and $aq - bp = 0$, then the two equations of [10.(3)] are the same and there are infinitely many centers (two parallel lines).

If $b^2 - ac \neq 0$, then there is a unique center.

12. Examples:

(i) Find the center of

$$52x^2 - 72xy + 73y^2 + 8x - 294y - 1167 = 0 .$$

Since $(52)(73) - (-36)^2 \neq 0$, a center exists. The equation can be written as

$$(x \quad y \quad 1)\begin{pmatrix} 52 & -36 & 4 \\ -36 & 73 & -147 \\ 4 & -147 & -1167 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

The center is the solution of

$$(x \quad y \quad 1)\begin{pmatrix} 52 & -36 \\ -36 & 73 \\ 4 & -147 \end{pmatrix} = 0,$$

or

$$\begin{cases} 52x - 36y + 4 = 0 \\ -36x + 73y - 147 = 0 . \end{cases}$$

The center is (2, 3).

(ii) Find the center of

$$16x^2 + 24xy + 9y^2 + 150x - 200y - 1000 = 0 .$$

Since $(12)^2 - (16)(9) = 0$, there is either no center or infinitely many centers. We have

$$(x \quad y \quad 1)\begin{pmatrix} 16 & 12 & 75 \\ 12 & 9 & -100 \\ 75 & -100 & -1000 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 .$$

But $p/q \neq 16/12$, i.e., $aq-pb \neq 0$. Therefore there is no center.

(iii) Find the center of

$$64x^2-96xy+36y^2+480x-360y+675=0.$$

We see that $(64)(36)-(-48)^2=0$ and $a/b=-64/48=p/q=-240/180$, i.e., $aq-pb=0$. Therefore there are infinitely many centers. We expect the conic to degenerate into two lines parallel to

$$64x-48y+240=0, \text{ or } 4x-3y+15=0.$$

We shall get them as follows. The two lines must be

$$8x-6y+h=0 \quad \text{and} \quad 8x-6y+k=0.$$

Writing

$$(8x-6y+h)(8x-6y+k) \equiv 64x^2-96xy+36y^2+480x-360y+675=0,$$

we obtain

$$\begin{cases} h+k=60 \\ hk=675, \end{cases}$$

and h and k are the roots of $x^2-60x+675=0$ which are $h=45$ and $k=15$. Thus the conic is

$$(8x-6y+45)(8x-6y+15)=0.$$

13. Change of coordinate system to the most convenient position: If a conic section has a center we translate the origin to the center. This translation changes the conic into the form

$$ax^2+2bxy+cy^2=l$$

because the origin becomes the center of symmetry. Now by a suitable rotation we change this to the form $\lambda x^2+\mu y^2=l$. This is sometimes called the canonical form.

14. Change to canonical form: Let a conic be of the form

(1) $$ax^2+2bxy+cy^2=l, \quad b\neq 0.$$

Then (1) can be written as

$$(x \;\; y)\begin{pmatrix} a & b \\ b & c\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=(l).$$

Now we know that the equations of a rotation of the coordinate system are

$$\begin{cases} x=x_1\cos\theta-y_1\sin\theta \\ y=x_1\sin\theta+y_1\sin\theta. \end{cases}$$

This in matrix form is

$$(x \;\; y)=(x_1 \;\; y_1)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Let $S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and $X = (x \quad y)$ and $X_1 = (x_1 \quad y_1)$. Then we see that

$$SS' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = S'S.$$

Therefore

$$XS' = (x \quad y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = X_1SS' = X_1I = (x_1 \quad y_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x_1 \quad y_1).$$

We also see that $IA = AI = A$, where A is any matrix. Now (1) can be written as

(2) $$(l) = XAX' = XIAIX' = XS'SAS'SX' = X_1SAS'X_1'.$$

Now the problem becomes that of finding S such that

$$SAS' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = B.$$

This problem happens to have a solution, and we are going to solve it. We see that $SAS' = B$ can be written as $SAS'S = BS$, or $SA = BS$, or

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

This equation gives

$$\begin{cases} a\cos\theta + b\sin\theta = \lambda\cos\theta \\ b\cos\theta + c\sin\theta = \lambda\sin\theta, \end{cases} \quad \text{and} \quad \begin{cases} -a\sin\theta + b\cos\theta = -\mu\sin\theta \\ -b\sin\theta + c\cos\theta = \mu\cos\theta \end{cases}.$$

These equations are written as

(3) $$\begin{cases} (a-\lambda)\cos\theta + b\sin\theta = 0 \\ b\cos\theta + (c-\lambda)\sin\theta = 0, \end{cases}$$
$$\begin{cases} (a-\mu)\sin\theta - b\cos\theta = 0 \\ -b\sin\theta + (c-\mu)\cos\theta = 0. \end{cases}$$

Since $\sin\theta$ and $\cos\theta$ can not be zero at the same time we must have

$$(a-\lambda)(c-\lambda) - b^2 = 0, \quad \text{or} \quad \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0,$$

and

$$(a-\mu)(c-\mu) - b^2 = 0, \quad \text{or} \quad \begin{vmatrix} a-\mu & b \\ b & c-\mu \end{vmatrix} = 0.$$

We observe that λ and μ must satisfy the same equation, i.e.,

(4) $$\begin{vmatrix} a-x & b \\ b & c-x \end{vmatrix} = 0, \quad \text{or} \quad x^2 - (a+c)x - b^2 + ac = 0.$$

This equation always has real roots because its discriminant is

$$(a+c)^2 - 4(b^2-ac) = (a-c)^2 + 4b^2 > 0 .$$

We also see that $\lambda \neq \mu$; otherwise $a = c$ and $b = 0$ which implies that (1) is a circle. Obtaining λ and μ we substitute in (2) and we get

$$X_1 SAS' X_1' = (x_1 \ \ y_1)\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix} = (l), \quad \text{or} \quad \lambda x_1^2 + \mu y_1^2 = l .$$

Indeed there are two rotations which change (1) to canonical form. Thus the other possibility is $\mu x_1^2 + \lambda y_1^2 = l$.

Now we did not need to find the rotations of coordinate system. But in case the rotations are desired we use equations (3) and find the rotations.

15. Example: Let $2x^2 + 4xy + 5y^2 = 1$ be the conic for which the coordinate system is to be rotated in order to eliminate the xy term. Clearly

$$1 = (x \ \ y)\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

is the same conic section. Thus [(14.(4)] will be

$$\begin{vmatrix}2-x & 2\\ 2 & 5-x\end{vmatrix} = 0,$$

and we get $\lambda = 1$, $\mu = 6$. We have either

$$x_1^2 + 6y_1^2 = 1, \quad \text{or} \quad 6x_1^2 + y_1^2 = 1 .$$

Now suppose we want to find the rotations. Equations [14.(3)] will be

$$\begin{cases}(2-1)\cos\theta + 2\sin\theta = 0\\ 2\cos\theta + (5-1)\sin\theta = 0,\end{cases}$$

$$\begin{cases}(2-6)\sin\theta - 2\cos\theta = 0\\ -2\sin\theta + (5-6)\cos\theta = 0 .\end{cases}$$

These equations are all equivalent since λ and μ satisfy [14.(4)]. We see that any of these equations reduces to

$$\cos\theta + 2\sin\theta = 0, \quad \text{or} \quad \tan\theta = -1/2 .$$

Since we are interested in $\sin\theta$ and $\cos\theta$ we observe that $\cos^2\theta = \dfrac{1}{1+\tan^2\theta} = 4/5$, and since $-\pi/2 \leqq \theta \leqq \pi/2$ we have $\cos\theta = 2\sqrt{5}/5$ and $\sin\theta = -\sqrt{5}/5$. Therefore the rotation which changes the conic into $x^2 + 6y^2 = 1$ is

$$(x \ \ y) = (x_1 \ \ y_1)\begin{pmatrix}2\sqrt{5}/5 & -\sqrt{5}/5\\ \sqrt{5}/5 & 2\sqrt{5}/5\end{pmatrix},$$

or

$$\begin{cases}x = (2\sqrt{5}/5)x_1 + (\sqrt{5}/5)y_1\\ y = (-\sqrt{5}/5)x_1 + (2\sqrt{5}/5)y_1 .\end{cases}$$

Now for the other rotation we write $\lambda = 6$ and $\mu = 1$. It is sufficient to use only

$$(2-6)\cos\theta + 2\sin\theta = 0, \quad \text{or} \quad -2\cos\theta + \sin\theta = 0, \quad \text{or} \quad \tan\theta = 2.$$

Comparing this with the previous case we see that one rotation takes the x-axis into a direction perpendicular to the direction into which the other rotation takes the x-axis.

Now $\cos^2\theta = \dfrac{1}{1+\tan^2\theta} = 1/5$ and as in the previous case, $-\pi/2 \leq \theta \leq \pi/2$ and we have

$$\cos\theta = \sqrt{5}/5 \quad \text{and} \quad \sin\theta = 2\sqrt{5}/5.$$

Thus the rotation which changes the conic into $6x_1^2 + y_1^2 = 1$ is

$$(x \quad y) = (x_1 \quad y_1)\begin{pmatrix} \sqrt{5}/5 & 2\sqrt{5}/5 \\ -2\sqrt{5}/5 & \sqrt{5}/5 \end{pmatrix}.$$

16. Example: (Reduction of a conic with center); By a translation and rotation simplify

$$52x^2 - 72xy + 73y^2 + 8x - 29y - 1167 = 0.$$

The reader may see that the center exists. We write the equation as

$$(x \quad y \quad 1)\begin{pmatrix} 52 & -36 & 4 \\ -36 & 73 & -147 \\ 4 & -147 & -1167 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

The center is obtained from

$$(x \quad y \quad 1)\begin{pmatrix} 52 & -36 \\ -36 & 73 \\ 4 & -147 \end{pmatrix} = 0,$$

or

$$\begin{cases} 52x - 36y + 4 = 0 \\ -36x + 73y - 147 = 0. \end{cases}$$

The solution is (2, 3). Therefore

$$\begin{cases} x = x_1 + 2 \\ y = y_1 + 3 \end{cases}$$

is the translation which puts the origin at the center. We observe that the substitution for x and y does not change the form of the second degree terms and the first degree terms will disappear. We have to compute only the new constant term. This constant is found as follows:

$$52(2^2) - 72(2)(3) + 73(3^2) + 8(2) - 294(3) - 1167 = -1600.$$

Since the left side of the equation equals the constant term when $x_1 = y_1 = 0$,

$$52x_1^2 - 72x_1y_1 + 73y_1^2 = 1600 .$$

Now we proceed as in [15.].

$$\begin{vmatrix} 52-x & -36 \\ -36 & 73-x \end{vmatrix} = 0$$

gives $\lambda = 100$ and $\mu = 25$. Thus either

$$100x_2^2 + 25y_2^2 = 1600 , \text{ or } 25x_2^2 + 100y_2^2 = 1600 ,$$

which reduces to

$$4x_2^2 + y_2^2 = 64, \text{ and } x_2^2 + 4y_2^2 = 64 \text{ respectively}.$$

We leave it to the reader to find the rotations if desired.

17. Reduction when no center exists : Let the conic be

$$ax^2 + 2bxy + cy^2 + 2px + 2qy + r = 0 \quad \text{and} \quad b^2 - ac = 0 .$$

Note that there might be infinitely many centers [see 11.II]. But we treat the problem in the same way. We write the equation as

(1) $$(x \;\; y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-2px - 2qy - r) .$$

Now we use a rotation to change the left side to the form

$$(x_1 \;\; y_1)\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} .$$

We know that λ and μ are the roots of $x^2 - (a+c)x - (b^2 - ac) = 0$ [see 14.(4)]. But using the relations between the roots and coefficients of this equation, we see that $\lambda + \mu = a + c$ and $\lambda\mu = b^2 - ac = 0$. Thus either $\lambda = 0$ or $\mu = 0$. Suppose $\mu = 0$ and $\lambda \neq 0$. This is possible since $\mu = 0$ and $\lambda = 0$ imply that $a + c = 0$ and thus $b^2 - ac = b^2 + a^2 = 0$ which means $a = 0$, $b = 0$ and it is of no interest. Therefore we have

$$\lambda = a + c \quad \text{and} \quad \mu = 0$$

and (1) will be

$$(x_1 \;\; y_1)\begin{pmatrix} a+c & 0 \\ 0 & 0 \end{pmatrix} x_1 \;\; = [f(x_1, y_1)] .$$

In order to determine $f(x_1, y_1)$ we have to find the equations of rotation. We use [14.(3)], i.e.,

$$a \sin\theta - b\cos\theta = 0 , \text{ or } \tan\theta = b/a . \text{ Then } \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{a^2}{a^2+b^2} .$$

But $b^2 - ac = 0$. Thus

$$\cos^2\theta = \frac{a^2}{a^2 + ac} = \frac{a}{a+c} .$$

Since $-\pi/2 \leqq \theta \leqq \pi/2$ we have

$$\cos\theta = \sqrt{\frac{a}{a+c}} .$$

We observe that $b^2 = ac > 0$ implies that a and c have the same sign and this shows that $a/(a+c) > 0$. Now $\sin\theta = \dfrac{b}{a}\cos\theta$ or $\sin\theta = \dfrac{b}{a}\sqrt{\dfrac{a}{a+c}}$.

Let us study different cases:

I. Let $b > 0$. Then $b = \sqrt{ac}$ and $\sin\theta = \dfrac{\sqrt{ac}}{a}\sqrt{\dfrac{a}{a+c}}$.

(i) If $a > 0$, then we have $\sin\theta = \sqrt{\dfrac{c}{a+c}}$.

(ii) If $a < 0$, then

$$\sin\theta = \frac{\sqrt{ac}}{-(-a)}\sqrt{\frac{a}{a+c}} = -\sqrt{\frac{(ac)(a)}{a^2(a+c)}} = -\sqrt{\frac{c}{a+c}}.$$

II. Let $b < 0$. Then $b = -\sqrt{ac}$ and $\sin\theta = \dfrac{-\sqrt{ac}}{a}\sqrt{\dfrac{a}{a+c}}$.

(ii) If $a > 0$, then we have $\sin\theta = -\sqrt{\dfrac{c}{a+c}}$.

(ii) If $a < 0$, then we have $\sin\theta = \dfrac{-\sqrt{ac}}{-(-a)}\sqrt{\dfrac{(ac)(a)}{a^2(a+c)}} = \sqrt{\dfrac{c}{a+c}}$. The result is

$$\cos\theta = \sqrt{\frac{a}{a+c}}, \quad \sin\theta = \pm\sqrt{\frac{c}{a+c}},$$

and the sign for the latter equation is the same as the sign of ab.

18. Example: Simplify

$$16x^2 + 24xy + 9y^2 + 150x - 200y - 1000 = 0$$

by a rotation and translation of coordinates.

We see that $\cos\theta = \sqrt{\dfrac{16}{9+16}} = \dfrac{4}{5}$, $\sin\theta = \sqrt{\dfrac{9}{9+16}} = \dfrac{3}{5}$. Thus the rotation is

$$\begin{cases} x = 4/5\, x_1 - 3/5\, y_1 \\ y = 3/5\, x_1 + 4/5\, y_1, \end{cases}$$

and $\lambda = 16 + 9 = 25$, $\mu = 0$. Thus we have

$$25x_1^2 + 150(4/5\, x_1 - 3/5\, y_1) - 200(3/5\, x_1 + 4/5\, y_1) - 1000 = 0,$$

$$\text{or}\quad 25x_1^2 - 250y_1 - 1000 = 0, \quad \text{or}\quad x_1^2 = 10(y_1 + 4).$$

By a translation of coordinates we get

$$x_2^2 = 10y_2.$$

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POLYNOMIALS AND FUNCTIONS

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1. Introduction:

The purpose of this paper is to clarify the nature of a variable and a function. This will be done by starting with the definition of a polynomial over a set, and defining the concept of a variable and function in terms of procedural rules.

2. Polynomials:

Let us consider a set S containing elements a_i ($i = 1, 2, 3, \cdots$) and operations 'o' and '$\square$'. We take any symbol 'x' and any other symbol '+' which are to be used as follows:

2.1. The symbol x will be juxtaposed to any element a_i on the right of a_i, to form an expression we will call a monomial.

2.2 If x is repeated after a given a_i then the number of x's in the monomial may be indicated by writing a superscript to x.

Example 2.3. a_1x, $a_{15}xx$, $a_{12}xxx$, etc. are monomials.

$a_{15}xx$ may be written as $a_{15}x^2$.

$a_{12}xxx$ may be written as $a_{12}x^3$.

The sub-scripts on the elements of S are solely for the purpose of distinguishing the elements of S.

2.4. The symbol '+' occurs only between monomials. For example:

$$a_1x^3 + a_{120}x^{57}.$$

2.5. Definition of a polynomial over a set S. A polynomial in a symbol x over a set S, is an expression of the form

$$a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

where $a_0, a_1, \cdots, a_n \in S$.

3. Comments: 3.1. x^0 is usually not indicated.

3.2. Since x is not necessarily an element of S the right hand juxtaposition of x to an element of S cannot be taken to indicate the multiplication of the element of S by the x. a_ix^i is not an element in S, even if S happens to be closed.

3.3. The + is neither the o nor the $\square$ of S. Therefore the polynomial in x over S cannot be identified with some element of S even if S happens to be closed.

3.4. We can form a polynomial in a_2 over S, $a_2 \in S$, as follows

$$a_0 + a_1a_2 + a_2a_2^2 + a_3a_2^3 + \cdots + a_na_2^n$$

and this will not be an element in S either.

4. Operations with polynomials in x over S.

Definition 4.1. A polynomial in x over a set S of elements $\{s_1, s_2, \cdots, s_n, \cdots\}$ will be denoted by $S(x)$.

$$S(x) = s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots + s_nx^n$$

Definition 4.2. If $S_1(x)$ and $S_2(x)$ are polynomials in x over a set S such that

$$S_1(x) = s_0 + s_1x + s_2x^2 + \cdots + s_nx^n$$

$$S_2(x) = s_0' + s_1'x + s_2'x^2 + \cdots + s_n'x^n$$

then $S_1(x) \equiv S_2(x)$ if and only if $s_ix^i = s_i'x^i$.

Definition 4.3. If $S(x)$ is a polynomial in x over a set S, then the degree of $S(x)$ will be the largest n for which x^n has a coefficient $s_n \neq 0$.

Definition 4.4. The power of a monomial is the exponent of x.

It follows that $S_1(x) \equiv S_2(x)$ implies $s_i = s_i'$, and the degree of $S_1(x)$ = degree of $S_2(x)$.

Definition 4.5. If $S_1(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $S_2(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ are polynomials in x over a set S with operation $\square$ (addition) and $a_i \in S$, $b_i \in S$, then

$$S_1(x) \oplus S_2(x) = (a_0 \square b_0) + (a_1 \square b_1)x + (a_2 \square b_2)x^2 + \cdots + (a_n \square b_n)x^n.$$

Theorem 4.6. If $S_1(x)$ and $S_2(x)$ are polynomials in x over a set S with operation $\square$ defined in S, then $S_1(x) \oplus S_2(x)$ is also a polynomial in x over S.

Proof: By definition

$$S_1(x) \oplus S_2(x) = (a_0 \square b_0) + (a_1 \square b_1)x + (a_2 \square b_2)x^2 + \cdots + (a_n \square b_n)x^n.$$

But since a_i and $b_i \in S$ (and we assume S closed with respect to $\square$)

$$a_i \square b_i \in S. \text{ Let } a_i \square b_i = c_i \in S$$

Therefore $S_i(x) \oplus S_2(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$.

5. The generalized Polynomial: Since nothing is said about the nature of the indeterminate x, we can form polynomials in y over a set S with operation $\square$, also. However, we would not necessarily want a polynomial in x over S to be distinct from every polynomial in y over S. To justify treating two polynomials as the same we cannot use a definition but will need to make use either of a procedural rule, or a rule of replacement.

Rule of Replacement 1. If $S(x)$ is a polynomial in x over a set S with operation $\square$ defined, then if the symbol x is replaced everywhere it occurs by another symbol of the same kind, for example, y, the resultant polynomial

$S(y)$ will be equivalent to $S(x)$. E. g.

$$5+x^2+2x^3+x^4 \equiv 5+y^2+2y^3+y^4.$$

The same result may be achieved by

Procedural Rule 1. If $S(x)$ is a polynomial in x over the set S and $S(y)$ is a polynomial in y over the set S, with the corresponding coefficients of $S(x)$ and $S(y)$ equal, then we will proceed as if $S(x) \equiv S(y)$. The symbols x and y must be of the same kind.

E. g. $5+x^2+2x^3+x^4 \not\equiv 5+\pi^2+2\pi^3+\pi^4$ because 'x' and 'π' are not symbols of the same kind. If the two polynomials are taken over the set R of real numbers $\pi \in R$, but $x \notin R$.

The Rule of replacement evidently follows from the Procedural Rule.

Procedural Rules will be directives to behave in a certain fashion. The essence of the concept of function lies in these procedural rules, as we will show.

Theorem 5.1. If $S_1(x)$ and $S_1(y)$ are polynomials over S in x and y with corresponding coefficients equal.

$$S_1(x) \equiv S_1(y).$$

Proof: $S_1(y)$ differs from $S_1(x)$ only in the fact that whenever there is an x in $S_1(x)$, there is a y in $S_1(y)$. Therefore $S_1(x)$ and $S_1(y)$ have the same degrees and the same corresponding coefficients. By Procedural Rule 1, we are entitled to assert $S_1(x) \equiv S_1(y)$.

This suggests that we need to generalize the notion of the indeterminate so that various types of symbols can be used to replace it.

Definition 5.2. We adopt a meaningless symbol ζ. A polynomial in ζ over a set S with the operation $\square$ defined in S is an expression of the form

$$a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n.$$

Procedural Rule 2. In a polynomial $S(\zeta)$ over a set S, when the symbol ζ is replaced by any other symbol, the resultant expression will also be treated as a polynomial over S.

Definition 5.3. If $P(\zeta)$ and $Q(\Sigma)$ are two polynomials over a set S with operation defined in S, then $P(\zeta) \equiv Q(\Sigma)$ if and only if for every symbol w, $P(w) \equiv Q(w)$.

6. More comments.

If ζ is replaced by x, we obtain a polynomial in x over the set S.

If ζ is replaced by $\sqrt{2}$, we obtain a polynomial in $\sqrt{2}$ over the set S.

We have used $\square$ to denote the operation of "addition" in S. We establish

Procedural Rule 3. In a polynomial in x over a set S with operation $\square$, replace the $\square$ by + whenever it occurs.

Since $S_1(x)$ and $S_2(x)$ are not polynomials but names for polynomials, "$S_1(x)+S_2(x)$" is the name for the sum of the two polynomials.

Evidently the + between two monomials in a polynomial is not the same as the + which replaces the $\square$ in S. No confusion results however from the indiscriminate use of these two symbols.

7. Mappings.

Evidently for any replacement of ζ by some symbol w in $S(\zeta)$, we obtain a mapping $\theta: w \to S(w)$, where $S(\zeta)$ is a general polynomial over the set S with operation $+$.

Definition 7.1. If $S(\zeta)$ is a polynomial in ζ over a set S with operation + defined in S, then $S(\zeta)$ will be called a polynomial function over S.

A polynomial function over a set S defines a set of polynomials over S.

Procedural Rule 4. To obtain polynomials over a set S, we replace the meaningless symbol in $S(\zeta)$ by indeterminates.

Suppose we have a polynomial in x over a set S. Given $S(x) = a_0 + a_1x + + a_2x^2 + \cdots + a_nx^n$.

We consider what happens when we replace x by elements of S. Let $s_1 \epsilon S$. Then by our procedural rule we can write $S(s_1) = a_0 + a_1s_1 + a_2s^2 + \cdots + a_ns_1^n$.

$S(s_1)$ is not yet an element of S because although $a_i \epsilon S$ and $s_1 \epsilon S$, $a_is_1 \not\epsilon S$ since juxtaposition was not equated with multiplication in S.

If S is a set with only one operation (e.g. a group), then $S(s)$ cannot be an element of S.

If S is a set with two operations, addition and multiplication, then we can adopt a

Procedural Rule 5. If $S(x)$ is a polynomial in x over a set S with addition and multiplication in S defined, then we can identify the + between monomials with addition in S and juxtaposition with multiplication in S, whenever x in $S(x)$ is replaced by an element s_i of S.

On the basis of P. R. 5, we can establish the following mapping

$$\theta_1 : s_i \to S(s_i) \quad \text{where} \quad s_i \epsilon S$$

Since we did not assume S to be closed $S(s_i)$ is not necessarily an element in S after the operations are performed.

If we add the condition that S be closed, then we have

$$\theta_2 : s_i \to S(s_i) \to s_j \quad \text{where} \quad s_i, s_j \epsilon S.$$

It follows that θ maps S into itself.

We can now define a function:

Definition 7.2. A function is a polynomial in x over a closed set S (with addition and multiplication defined in S), such that only elements of S are used to replace x; or S is mapped into itself by the polynomial when x is replaced by elements of S.

Definition 7.3. A variable in a polynomial in x over a set S (with addition and multiplication defined in S) is an indeterminate in the polynomial which may be replaced in accordance with procedural rules, by elements of S.

Boston College

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

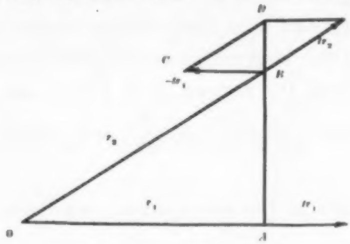
Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

RECEDING GALAXIES

Lindley J. Burton

Hubble proposed a generally accepted astronomical theory that galaxies recede from each other at a rate proportional to the distance between them. The red shift of the spectral lines supports this theory.

Here I wish to show with the help of vectors that if A and B each recede from 0 at a rate proportional to the distances $0A$ and $0B$, then B recedes from A at a rate proportional to AB in Euclidean space.



Let $\vec{0A} = \vec{r}_1$ and $v(A, 0) = t\vec{r}_1$, and $\vec{0B} = \vec{r}_2$ and $v(B, 0) = t\vec{r}_2$. Then $v(B, A)$ = resulting vector from $-t\vec{r}_1$ and $t\vec{r}_2$ at B. Clearly $\Delta A0B \sim \Delta BCD$ since $<0 = <C$ and the sides including these angles are proportional. Hence $|\vec{BD}| = t|\vec{AB}|$, and $<CBD = <0AB$, so that BD is an extension of AB. Thus $\vec{BD} = t\vec{AB}$, which proves the result with the same constant of proportionality.

This result means that there is *no unique* center from which the galaxies recede. Suppose that the centers of the galaxies are at the vertices of a grid infinite in extent and that the edge of the basic square of the grid is increasing in length at a fixed rate. Then the galaxies recede from each other at a rate proportional to their distance.

Lake Forest College

A NOTE ON SPACE CURVES

Major H. S. Subba Rao

Problem No. 248 appearing in Vol. 29, No. 1, September 1955, this Journal, proposed by Huseyin Demir, Turkey, can be extended to three dimensions.

It is well known that if Γ_1 and Γ_2 are two twisted curves in one-to-one correspondence such that the tangents at corresponding points are parallel then the principal normals and binormals at corresponding points are also parallel. The two curves are said to be deducible from each other by a Combesure transformation.

It can be stated as a theorem that the lines joining corresponding points on two twisted curves deducible from each other by a combesure transformation generate a developable surface and that the line joining the centres of curvature at corresponding points intersects the generator at a point on the edge of regression of the developable surface.

Proof: Let $\bar{r}_1 = f(s_1)$ and $\bar{r}_2 = \phi(s_2)$ be the vector equations to the two curves, $\bar{r}_1$ and $\bar{r}_2$ being the vectors to corresponding points, s_1 and s_2 the arc lengths measured along the curves. If $\bar{t}_1$ and $\bar{t}_2$ are the unit tangents at corresponding points then $\dfrac{d\bar{r}_1}{ds_1} = \bar{t}_1$ and $\dfrac{d\bar{r}_2}{ds_2} = \bar{t}_2$ and $\bar{t}_1 = \bar{t}_2$. If $\bar{n}_1, \bar{n}_2$ are the unit principal normals at the same points we have $\dfrac{d\bar{t}_1}{ds_1} = \dfrac{\bar{n}_1}{\rho_1}$ and $\dfrac{d\bar{t}_2}{ds_2} = \dfrac{\bar{n}_2}{\rho_2}$ where ρ_1 and ρ_2 are the radii of curvature of the two curves. Since $\dfrac{d\bar{t}_2}{ds_2} = \dfrac{d\bar{t}_1}{ds_1} \cdot \dfrac{ds_1}{ds_2}$ we get $\dfrac{\bar{n}_2}{\rho_2} = \dfrac{\bar{n}_1}{\rho_1} \cdot \dfrac{ds_1}{ds_2}$ showing that $\bar{n}_1 = \bar{n}_2$ and $\dfrac{ds_1}{ds_2} = \rho_1/\rho_2$. Using Serret-Frenet formulae it will be obvious that the binormals are also parallel.

Let $\bar{d}$ be the unit vector in the direction of the vector $\bar{r}_1 - \bar{r}_2$, that is, along the line joining the corresponding points on the two curves. If $\bar{d}' = \dfrac{d}{ds_1}(\bar{d})$, $a^2 = \bar{d}'^2$, $b = \bar{t}_1 \cdot d'$ we get by differentiating the vector equation

$$\iota\bar{d} = \bar{r}_1 - \bar{r}_2 \quad (\iota = |r_1 - r_2|)$$

$$\iota\bar{d}' + \iota'\bar{d} = \bar{t}_1 - \frac{d\bar{r}_2}{ds_2} \cdot \frac{ds_2}{ds_1} \quad \left(\iota' = \frac{d\iota}{ds_1}\right)$$

$$= \bar{t}_1(1-\frac{\rho_2}{\rho_1}), \text{ since } \bar{t}_2 = \bar{t}_1 .$$

Squaring both sides and noting that $\bar{d}\cdot\bar{d}' = 0$ we get

$$a^2\iota^2 + \iota'^2 = (1-\frac{\rho_2}{\rho_1})^2$$

Differentiating the equation

$$(\bar{r}_1 - \bar{r}_2)^2 = \iota^2$$

we get

$$(\bar{r}_1 - \bar{r}_2)\cdot\bar{t}_1(1-\frac{\rho_2}{\rho_1}) = \iota\iota' \quad \text{or} \quad (1-\frac{\rho_2}{\rho_1})\cos\theta = \iota'$$

where θ is the angle between the vector $\bar{d}$ and $\bar{t}_1$.

Hence we get

$$a^2 = (1-\frac{\rho_2}{\rho_1})^2\frac{\sin^2\theta}{\iota^2}$$

Similarly we can show that

$$\bar{t}_1\cdot\bar{d}' = b = (1-\frac{\rho_2}{\rho_1})\frac{\sin^2\theta}{\iota}$$

It is well known from differential geometry that the condition for the ruled surface generated by the vector $\bar{r}_1 - \bar{r}_2$ to be a developable surface is that

$$b^2 = a^2\sin^2\theta$$

which is satisfied in this case.

Writing the equation to the generator $\bar{r}_1 - \bar{r}_2$ as

$$\bar{r} = \bar{r}_1 + k(\bar{r}_1 - \bar{r}_2)$$

where K is a scalar variable, we get the point of intersection of this generator with a neighbouring generator by solving for K from the equation

$$\frac{d\bar{r}_1}{ds_1} + K(\frac{d\bar{r}_1}{ds_1} - \frac{d\bar{r}_2}{ds_2}\cdot\frac{ds_2}{ds_1}) = \bar{t}_1 + K(t_1 - t_1\frac{\rho_2}{\rho_1}) = 0$$

and substituting in the equation to the generator. The vector to the point of intersection C, which is a point on the edge of regression of the developable surface, is given by

$$\bar{r} = \bar{r}_1 - \frac{\rho_1}{\rho_1 - \rho_2}(\bar{r}_1 - \bar{r}_2) = -\bar{r}_1 \frac{\rho_2}{\rho_1 - \rho_2} + \bar{r}_2 \frac{\rho_1}{\rho_1 - \rho_2}$$

Since the principal normals at corresponding points are parallel, the centres of curvature and the generator lie on a plane. If M_1, M_2 are corresponding points, C_1, C_2 the centres of curvature at these points then it is easy to see that the lines M_1M_2 and C_1C_2 intersect externally at a point C' such that

$$C'M_1 : C'M_2 = \rho_1 : \rho_2$$

Thus the vector C' is given by the same expression as for the vector to C. Hence C and C' coincide.

It will be seen that the problem proposed by Huseyin Demir can be obtained by unrolling the developable surface into a p`ane.

Defence Science Organisation
New Delhi
India

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India Ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

397. *Proposed by Leo Moser, University of Alberta.*

Show that the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}+\frac{1}{x_1\cdot x_2\cdot x_3\cdots x_n}=1$$

has at least one solution for every n.

398. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Determine the roots of the equations $x^2+y_1x+y_2=0$, $y^2+x_1y+x_2=0$ where the coefficients (real numbers) in one equation are the roots of the other.

399. *Proposed by Nathan Altshiller Court, University of Oklahoma.*

Prove that the feet of the perpendiculars dropped upon the sides of a triangle from their respective Simson poles form a cevian triangle. That is, the lines joining those points to the respectively opposite vertices are concurrent.

400. *Proposed by Walter B. Carver, Cornell University.*

Given a point, a circle, and any curve in a plane. Construct an equilateral triangle having a vertex on each of them.

401. *Proposed by John M. Howell, Los Angeles City College.*

Given a sequence of numbers related by $F(n) = aF(n-1)+bF(n-2)$, $F(0) = c$ and $F(1) = d$, where $n = 0, 1, 2, \cdots$ and a, b, c, and d are any real numbers. Find a general form for $F(n)$.

402. *Proposed by Chih-yi Wang, University of Minnesota.*

I. Show that if $|x| < 1$,

$$F(x) = \sum_{n=k}^{\infty} \binom{n}{k} x^n = x^k(1-x)^{k-1}.$$

II. Prove the same equality without using the binomial coefficients of negative arguments.

403. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given four lines, a_i $(i = 1, 2, 3, 4)$, determining a four space. Prove that if a_i meets a_{i+1} (mod 4), then all the given lines concur.

SOLUTIONS

Unorthodox Definitions

376. [May 1959] *Proposed by C. W. Trigg, Los Angeles City College.*

Identify the following unorthodox definitions: (1) the earthy or stony substance in which an ore or other mineral is bedded, (2) fully sufficient, (3) the four brightest components of θ Orionis, (4) type of speech fancifully exaggerated, (5) oddity, (6) a Franciscan friar, (7) a deposit formed in a liquid vegetable extract, (8) a large molding of convex profile, (9) fancied, (10) the series of air bubbles made by the breath from an otter under water, (11) a drain to carry off filthy water, (12) one sixteenth of a fluid drachm, (13) scheme, (14) depicted as walking, (15) absent-minded, (16) seekers, (17) complete, (18) invalid, (19) an interpreter of music.

Solution by the proposer. (1) matrix, (2) abundant, (3) trapezium, (4) hyperbolic, (5) eccentricity, (6) Minor, (7) apothem, (8) torus, (9) imaginary, (10) chain, (11) sink, (12) minimum, (13) angle, (14) gradient, (15) abstract, (16) zetetics, (17) integral, (18) null, (19) exponent. Note the acrostic formed by the initial letters of the terms.

Also solved by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts.

A Binomial Congruence

377. [May 1959] *Proposed by J. M. Gandhi, Thapar Polytechnic, Patiala, India.*

Prove that

$$\sum_{i=0}^{3n+1} \binom{6n+2}{2i} 3^i \equiv 0,\ \ 2^{3n+1},\ \ -2^{3n+1},\ \ [\bmod 2^{3n+2}]$$

when n is of the form $2m$, $4m+3$ or $4m-1$ respectively.

Solution by Leonard Carlitz, Duke University. Put

$$S = \sum_{r=0}^{3n+1} \binom{6n+2}{2r} 3^r .$$

Then by the binomial theorem

$$2S = (1+\sqrt{3})^{6n+2} + (1-\sqrt{3})^{6n+2}.$$

Let $\epsilon = 2+\sqrt{3}$, $\epsilon' = 2-\sqrt{3}$, so that

$$(1+\sqrt{3})^2 = 2\epsilon, \quad (1-\sqrt{3})^2 = 2\epsilon'$$

and therefore

$$2S = 2^{3n+1}(\epsilon^{3n+1} + \epsilon'^{3n+1}).$$

Now if n is odd

$$\tfrac{1}{2}(\epsilon^{3n-1} + \epsilon'^{3n+1}) = \sum_{r=0}^{\frac{1}{2}(3n+1)} \binom{3n+1}{2r} 2^{3n+1-2r} 3^r \equiv 3^{\frac{1}{2}(3n+1)} \equiv (-1)^{\frac{1}{2}(n-1)} \pmod{4},$$

so that

$$(1) \qquad S = (-1)^{\frac{1}{2}(n-1)} 2^{3n+1} \pmod{2^{3n+3}} \quad (n \text{ odd}).$$

If n is even

$$\tfrac{1}{2}(\epsilon^{3n+1} + \epsilon'^{3n+1}) = \sum_{2r \leqq 3n} \binom{3n+1}{2r+1} 2^{2r+1} 3^{3n-2r} \equiv 2(6n+1)3^{3n} \equiv 4n+2 \pmod{8},$$

so that

$$(2) \qquad S \equiv 2^{3n+2}(2n+1) \pmod{2^{3n+4}} \quad (n \text{ even}).$$

Thus (1) and (2) furnish the desired result and indeed a little more.

Also solved by Melvin Hochster and Jeff Cheeger (jointly) and the proposer.

Sum of a Series

378. [May 1959] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

Let a_0, a_1 be arbitrary and $n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}$ for $n > 0$. Find $S = \sum_0^\infty a_n$.

I. *Solution by William M. Sanders, Mississippi Southern College.* Since $a_2 = a_0/2$ and $a_3 = a_0/6$, a_n depends only on a_0 for $n \geqq 2$. By induction, $a_n = a_0/n!$. Consequently

$$S = \sum_0^\infty a_n = a_0 + a_1 + \sum_2^\infty \frac{a_0}{n!} = a_1 + a_0(e-1).$$

II. *Solution by J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University*. From the given recurrence relation, we have the identity

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}Z^n = \sum_{n=1}^{\infty} n(n-1)a_n Z^n - \sum_{n=1}^{\infty}(n-2)a_{n-1}Z^n$$

which is equivalent to

$$Z\frac{d^2}{dZ^2}\left[\sum_{n=1}^{\infty} a_{n+1}Z^{n+1}\right] = Z^2\frac{d^2}{dZ^2}\left[\sum_{n=1}^{\infty} a_n Z^n\right] - Z^3\frac{d}{dZ}\left[\sum_{n=1}^{\infty} a_{n-1}Z^{n-2}\right].$$

Letting $I(Z) = \sum_{n=0}^{\infty} a_n Z^n$, this latter identity may be written

$$Z\frac{d^2}{dZ^2}\left[I(Z) - a_1 Z - a_0\right] = Z^2\frac{d^2}{dZ^2}\left[I(Z) - a_0\right] - Z^3\frac{d}{dZ}\left[\frac{I(Z)}{Z}\right]$$

or,

$$(1-Z)\frac{d^2I}{dZ^2} = I(Z) - Z\frac{dI}{dZ}.$$

It is obvious that $AZ + Be^Z$ is the general solution of this equation. Further, since $I(0) = a_0$ and $\left.\frac{dI}{dZ}\right|_{Z=0} = a_1$,

$$I(Z) = (a_1 - a_0)Z + a_0 e^Z.$$

The desired sum is equal to $I(1) = a_1 + (e-1)a_0$.

III. *Solution by Victor Ch'iu, Kent State University, Ohio.*

$$\begin{aligned}
n(n+1)a_{n+1} &= n(n-1)a_n - (n-2)a_{n-1}\\
(n-1)na_n &= (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}\\
\cdots &= \cdots - \cdots\\
\cdots &= \cdots - \cdots\\
4\cdot 5a_5 &= 4\cdot 3a_4 - 2a_3\\
3\cdot 4a_4 &= 3\cdot 2a_3 - a_2\\
2\cdot 3a_3 &= 2\cdot 1a_2 - 0a_1\\
+\quad 1\cdot 2a_2 &= 1\cdot 0a_1 - (-1)a_0\\
\hline
\end{aligned}$$

$$n(n+1)a_{n+1} = a_0 - a_2 2a_3 - 3a_4 - 4a_5 - \cdots - (n-2)a_{n-1}$$

Hence,

$$a_2 = a_0(2!)^{-1}$$

$$a_3 = a_0(3!)^{-1}$$

$$a_4 = a_0(4!)^{-1}$$

$$\cdots$$

$$a_n = a_0(n!)^{-1}$$

Hence,

$$\begin{aligned} S &= a_0 + a_1 + a_2 + a_3 + a_4 + \cdots \\ &= a_1 + a_0\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!}\right) \\ &= a_1 + a_0(e-1) \end{aligned}$$

If we assume $a_0 = a_1$, then $S = ea_0$.

Also solved by George M. Bergman, Stuyvesant High School, New York; D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Leonard Carlitz, Duke University (two solutions); Jeff Cheeger, Erasmus Hall High School, Brooklyn, New York; Melvin Hochster, Stuyvesant High School, New York; Philip M. Mills, Virginia Polytechnic Institute; Paul Stygar, Norwich Free Academy, Connecticut; William B. Teachout, Jr., Memphis, Tennessee; George B. Thomas, Jr., Concord, Massachusetts; Chih-yi Wang, University of Minnesota; and the proposer.

The Duel

379. [May 1959] *Proposed by J. M. Howell, Los Angeles City College.*

Two men fight a duel. They both fire at a given signal. If both are alive, they fire again at a given signal, and repeat the process until at least one of them is dead. If the probability that A kills B on any round is p, and the probability that B kills A on any round is r, find the probability that A is alive, B is alive or neither is alive after: 1) n rounds, 2) after an infinite number of rounds.

I. *Solution by Glenn D. James, Los Angeles City College.* If one knows that any one round is about to occur, the probabilities at the end of that round are:

$$\begin{aligned} P(dd) &= P(\text{both dead}) = pr \\ P(aa) &= P(\text{both alive}) = (1-p)(1-r) \\ P(da) &= P(b \text{ kills } a) = r(1-p) \end{aligned}$$

$$P(ad) = P(a \text{ kills } b) = p(1-r)$$

For convenience let R be the probability: $P(dd)+P(da)+P(ad)$. Then the probability of death at the end of n trials is given on either side of the equation:

$$kR = 1 - P^n(aa)$$

The desired probabilities are $kP(ad)$, $kP(da)$, and $kP(dd)$, respectively. For n infinite, k is $1/R$.

II. *Solution by F. D. Parker, University of Alaska.* Let $P^1 = 1-p$, $r^1 = 1-r$, $P^n(AB)$ be the probability that both A and B are alive after the nth round, $P^n(AB^1)$ the probability that only A is alive after the nth round, etc. It is not hard to show by means of a tree or the appropriate stochastic matrix that $P^n(AB) = (p^1r^1)^n$, $P^n(AB^1) = \dfrac{pr^1[1-(p^1r^1)^n]}{1-p^1r^1}$, $P^n(A^1B) = \dfrac{p^1r[1-(p^1r^1)^n]}{1-p^1r^1}$, and $P(A^1B^1) = \dfrac{pr[1-(p^1r^1)^n]}{1-p^1r^1}$. Then we have $P^n(A) = \dfrac{pr^1+r(p^1r^1)^n}{1-p^1r^1}$, $P^n(B) = \dfrac{p^1r+p(p^1r^1)^n}{1-p^1r^1}$, and $P^n(A^1B^1) = \dfrac{pr[1-(p^1r^1)^n]}{1-p^1r^1}$. After an infinite number of rounds, these probabilities become $\dfrac{pr^1}{1-p^1r^1}$, $\dfrac{p^1r}{1-p^1r^1}$ and $\dfrac{pr}{1-p^1r^1}$, respectively.

III. *Solution by the proposer.* Consider:

State 1 – Both alive

State 2 – A only alive

State 3 – B only alive

State 4 – Neither alive

Then

$$A = \begin{pmatrix} qs & 0 & 0 & 0 \\ ps & 1 & 0 & 0 \\ qr & 0 & 1 & 0 \\ pr & 0 & 0 & 1 \end{pmatrix}$$

is a matrix such that if x_i is a column vector representing the relative probabilities at step i, AX_i will represent probabilities at step $i+1$. Then if we take $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_n = A^n X_0$. If we modify A as follows:

$$A = \begin{pmatrix} qs & 0 & 0 & 0 \\ ps & 1-\epsilon & 0 & 0 \\ qr & 0 & 1-\delta & 0 \\ pr & \epsilon & \delta & 1 \end{pmatrix}$$

we can find eigenvalues which are solutions of $|A-\lambda I| = 0$.

$$\lambda_1 = 1, \quad \lambda_2 = 1-\delta, \quad \lambda_3 = 1-\epsilon, \quad \lambda_4 = qs$$

then we can find eigenvectors which are solutions of:

$$AQ_i = \lambda_i Q_i$$

$$Q_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} qs-1 \\ ps \\ qr \\ pr \end{pmatrix}$$

then we can find constants C_i such that

$$X_0 = \sum C_i Q_i$$

$$C_1 = 1, \quad C_2 = \frac{qr}{1-qs}, \quad C_3 = \frac{ps}{1-qs}, \quad C_4 = \frac{-1}{1-qs}$$

then

$$X_n = A^n S_0 = \sum C_i \lambda_i^n Q_i$$

or

$$X_n = \frac{1}{1-qs} \begin{pmatrix} (1-qs)(qs)^n \\ ps - ps(qs)^n \\ qr - qr(qs)^n \\ 1 - qs - qr - ps - pr(qs)^n \end{pmatrix}$$

$$X_\infty = \lim_{n\to\infty} X_n = \frac{1}{1-qs} \begin{pmatrix} 0 \\ ps \\ qr \\ 1 - qs - qr - ps \end{pmatrix} = \frac{1}{1-qs} \begin{pmatrix} 0 \\ ps \\ qr \\ pr \end{pmatrix}$$

The results can be expressed a little easier if we let

$$K = 1 - qs, \quad K_n = 1 - (qs)^n$$

then

$$X_n = \frac{1}{K} \begin{pmatrix} k(qs)^n \\ psk_n \\ qrk_n \\ prk_n \end{pmatrix}, \quad X_\infty = \frac{1}{K} \begin{pmatrix} 0 \\ ps \\ qr \\ pr \end{pmatrix}$$

Also solved by George M. Bergman, Stuyvesant High School, New York; Jeff Cheeger, Erasmus Hall High School, Brooklyn, New York; and Melvin Hochster, Stuyvesant High School, New York. Two incorrect solutions were received.

A System of Equations

380. [May 1959] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Solve the system of equations

$$x(z-a)+u(x+u)=0 \tag{1}$$

$$y(x-b)+u(y+u)=0 \tag{2}$$

$$z(y-c)+u(z+u)=0 \tag{3}$$

where $abc \neq 0$ and $a^{-1}+b^{-1}+c^{-1}=u^{-1}$.

Solution by Chih-yi Wang, University of Minnesota. By performing the operations multiply (1) by y, multiply (2) by z, multiply (3) by x and applying (2), (3), (1) respectively we get

$$xyz-aby+auy+au^2+buy-u^3=0 \tag{4}$$

$$xyz-bcz+buz+bu^2+cuz-u^3=0 \tag{5}$$

$$xyz-cax+cux+cu^2+aux-u^3=0 \tag{6}$$

By performing the operations (4)−(5), (5)−(6), (6)−(4) we get

$$(au+bu-ab)y+(bc-bu-cu)z=(b-a)u^2 \tag{7}$$

$$(ca-cu-au)x+(cu+bu-bc)z=(c-b)u^2 \tag{8}$$

$$(au+cu-ca)x+(ab-au-by)y=(a-c)u^2 \tag{9}$$

Since the augmented matrix of (7), (8), (9) is of rank 2, we can calculate two variables in terms of the third, so we get

$$y=\frac{c^2}{a^2}z-\frac{c(b-a)}{ab}u \tag{10}$$

$$x=\frac{b^2}{a^2}z+\frac{b(c-b)}{ca}u \tag{11}$$

By substituting (10) into (3) we get, after simplification,

$$\left(\frac{c}{a}z-u\right)^2=0$$

whence by aid of (10) and (11), we obtain

$$x=(b/a)u, \quad y=(c/b)u, \quad z=(a/c)u.$$

Note that we have used the relations $abc \neq 0$, $a^{-1}+b^{-1}+c^{-1} = u^{-1}$ whenever necessary.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Victor Ch'in, Kent State University, Kent, Ohio; Melvin Hochster, Stuyvesant High School, New York; and the proposer.

Spheres About a Tetrahedron

381. [May 1959] *Proposed by George M. Bergman and Melvin Hochster, Stuyvesant High School, New York.*

Show that if the circumsphere and the insphere of a tetrahedron are concentric, four spheres can be drawn each tangent to each edge of the tetrahedron, extended if necessary.

Solution by J. W. Clawson, Collegeville, Pennsylvania. Let A_1, A_2, A_3, A_4 be the vertices of the tetrahedron; a_{12} the length of edge A_1A_2; B_{12} the midpoint of that edge; O_1 the circumcenter of the face opposite A_1; and so on; O the circumcenter of the tetrahedron.

Obviously OO_1 is perpendicular to the face opposite A_1 and O_1B_{23} to the edge A_2A_3; and so on.

Now if O is also the in-center of the tetrahedron, $OO_1 = OO_2$. Hence $O_1B_{34} = O_2B_{34}$; triangle $O_1B_{34}A_3$ = triangle $O_2B_{34}A_3$ and $O_1A_3 = O_2A_3$. Thus the circumradii of the faces are all equal. It follows at once that face angles $A_3A_2A_4$, $A_3A_1A_4$ are equal and in general that face angles opposite a common edge in two of the faces are equal.

Consider triangle $A_1A_2A_3$. Let $<A_1 = \alpha$, $<A_2 = \beta$, then $<A_3 = \pi-\alpha-\beta$. Then in triangle $A_1A_2A_4$, $<A_4 = \pi-\alpha-\beta$, let $<A_2 = \gamma$, then $<A_1 = \alpha+\beta-\gamma$. In triangle $A_1A_3A_4$, $<A_4 = \beta$, $<A_3 = \gamma$, then $<A_1 = \pi-\beta-\gamma$. And in triangle $A_2A_3A_4$, $<A_2 = \pi-\beta-\gamma$, $<A_3 = \alpha+\beta-\gamma$, $<A_4 = \alpha$. Hence $\pi+2\alpha-2\gamma = \pi$, or $\alpha = \gamma$. It follows that all the faces are similar; and, since their circumradii are equal, that they are congruent. Thus opposite edges a_{12}, a_{34}; a_{13}, a_{24}; a_{14}, a_{23} are respectively equal. Note, however, the restriction that, since the sum of the face angles at any vertex is π, and since the sum of two of these face angles must be greater than the third, all of the face angles are acute. Thus a tetrahedron whose circumcenter and incenter coincide has its opposite edges equal, but its congruent faces must be acute-angled triangles.

Now, if a sphere touches the six edges of the tetrahedron, its sections by the four faces are circles inscribed or escribed to the triangular faces. Let the points of contact of the incircle to triangle $A_2A_3A_4$ be ${}_1I_{23}$, ${}_1I_{24}$, ${}_1I_{34}$, and those of the excircle opposite A_1 to triangle $A_1A_2A_3$ be ${}_4J_{12}$, ${}_4J_{13}$, ${}_4J_{23}$; and so on. Let s be the semi-perimeter of each of the congruent faces.

Then $A_{21}I_{23} = s = a_{34} = s - a_{12}$; and $A_{14}J_{12} = s$, $A_{24}J_{23} = s - a_{12}$. In this way we may show that the inscribed circle to any face touches any one of its sides at the same point at which the escribed circle to the other face containing that side touches that side; and that escribed circles to two faces which have an edge in common touch that edge produced in the same point. This proves the proposal.

Also solved by the proposers. N. A. Court, University of Oklahoma submitted two bibliographical notes:

a. If in a tetrahedron the circumcenter and the incenter coincide, each edge is equal to the opposite edge, and the tetrahedron is said to be isosceles (Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 97, art. 304).

b. With an isosceles tetrahedron four spheres may be associated each touching three edges internally and the remaining three edges externally (John Leech, Some Properties of the Isosceles Tetrahedron, *Mathematical Gazette*, London, vol. 34, no. 310 (December, 1950), p. 270, Theorem 8).

The Hypocycloid

382. [May 1959] *Proposed by C. N. Mills, Sioux Falls College, South Dakota.*

For the hypocycloid $x^{2/3}+y^{2/3} = a^{2/3}$, (α, β) is the center of curvature. Show that $\alpha+\beta = (x^{1/3}+y^{1/3})^3$.

Solution by D. A. Breault, Sylvania Electric Products Inc., Waltham, Massachusetts. Since we have,

$$y = (a^{2/3}-x^{2/3})^{3/2}, \tag{1}$$

we find from a book of tables and formulas that the center of curvature, (α, β), of $y = f(x)$ is given by,

$$\alpha = x - y'(1+y'^2)/y'', \tag{2}$$

$$\beta = y+(1+y'^2)/y'', \tag{3}$$

whence

$$\alpha+\beta = x+y+(1+y'^2)(1-y)/y''. \tag{4}$$

Calculation of y' and y'' from (1), and substitution into (4) gives the desired result:

$$\alpha+\beta = (x^{1/3}+y^{1/3})^3. \tag{5}$$

(Intermediate calculations have been omitted to avoid making the proof very much longer than it is now.)

Also solved by Norman Anning, Alhambra, California; Jeff Cheeger, Erasmus Hall High School, Brooklyn, New York; Victor Ch'in, Kent State University, Kent, Ohio; Melvin Hochster, Stuyvesant High School, New York;

F. D. Parker, University of Alaska; William M. Sanders, Mississippi Southern College; C. M. Sidlo, Framingham, Massachusetts; Chih-yi Wang, University of Minnesota; Ron Wilder, Grants Pass, Oregon; Dale Woods, State Teachers College, Kirksville, Missouri, and the proposers.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 263. Solve $\arctan(p/x)+\arctan(q/x)+\arctan(r/x) = \pi$ for x where p, q and r are fixed. [*Submitted by Melvin Hochster and Jeff Cheeger*]

Q 264. Evaluate $\displaystyle\sum_{m=1}^{M}\sum_{n=1}^{N}\min(m,n)$ for $M \geq N$. [*Submitted by M. S. Klamkin*]

Q 265. Show that $\displaystyle\sum_{p\ \text{prime}} 1/p$ diverges. [*Submitted by David L. Silverman*]

Answers

A 263. In the diagram we have

$\arctan(p/x)+\arctan(q/x)+\arctan(r/x) = \alpha+\beta+\gamma = (\pi/2)-(A/2)+(\pi/2)-(B/2)+(\pi/2)-(C/2) = (3\pi/2)-\pi/2 = \pi$. Thus x is the radius of the incircle of a triangle of sides $p+q$, $q+r$, and $r+p$ and has the value $\sqrt{pqr/(p+q+r)}$.

A 264. $$\sum_{m=1}^{M}\sum_{n=1}^{N}\min(m,n) = \sum_{n=1}^{N}\left(\sum_{m=1}^{n}(m) + \sum_{m=n+1}^{M}(n)\right) = \sum_{n=1}^{N}\left[\frac{n}{2}(n+1) + n(m-n)\right]$$

Since $\sum_{n=1}^{N} n^2 = \frac{n(n+1)(2n+1)}{6}$, we have our sum equal to $\frac{N(N+1)(3M-N+1)}{6}$.

A 265. The probability that a randomly selected integer has no prime divisor may be represented as a product of independent probabilities: $(1-1/2)(1-1/3)(1-1/5)\cdots$ Since one is the only integer that meets this criterion, we have $\Pi(1-p) = 0$. Since no factor is zero, the infinite product diverges to zero. But $\sum a_n$ converges or diverges as $\Pi(1-a_n)$ converges or diverges. Thus $\sum_{p \text{ prime}} (-1/p)$ and likewise $\sum_{p \text{ prime}} (1/p)$ diverges.

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 34. Twenty sheep are to be slaughtered in five days such that (1) only an odd number of sheep can be slaughtered in each day, and (2) no killing in any one day is ruled out since zero is naturally regarded as an even number. How can the schedule to slaughter be arranged? [*Submitted by Chih-yi Wang*]

T 35. Double the area of a square hole in a wall without increasing the height or width of the hole. [*Submitted by Lindley J. Burton*]

Solutions

S 34. The sum of five odd integers is odd. Therefore there is no solution.

S 35. The solution depends upon taking the original square hole in the wall at an angle of 45° with the horizontal. Then a square with sides horizontal and vertical lines through the original vertices will have twice the area yet retain the same height and width.

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